DUALITY

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Let $X$ and $Y$ be topological vector spaces and $T \in L(X,Y)$. The adjoint of $T$ is the linear map $T^* : Y^* \rightarrow X^*$ defined by $T^* f = f \circ T$.

**Proposition.** Let $X$ and $Y$ be topological vector spaces and $T \in L(X,Y)$. Then:

1. $T^*$ is weak*-continuous.
2. If $X$ and $Y$ are normed then $\|T^*\| = \|T\|$. 

If $X$ is a topological vector space, the annihilator of $A \subset X$ is

$$A^\perp := \{ f \in X^* : f(x) = 0 \text{ for all } x \in A \},$$

and the preannihilator of $B \subset X^*$ is

$$\perp B := \{ x \in X : f(x) = 0 \text{ for all } f \in B \}.$$

**Proposition.** Let $X$ and $Y$ be topological vector spaces and $T \in L(X,Y)$. Then

$$\ker T^* = (\operatorname{ran} T)^\perp,$$

and if $Y$ is locally convex, then we also have

$$\ker T = \perp (\operatorname{ran} T^*).$$

**Proposition.** If $Y$ is a subspace of a locally convex space, then $\perp (Y^\perp) = \overline{Y}$. 

**Corollary.** Let $X$ be a locally convex space, and let $Y$ be a subspace of $X^*$. Then $(\perp Y)^\perp = \overline{Y}^{\text{wk}*}$. 

**Proposition.** Let $X$ be a locally convex space, and let $Y$ be a closed subspace. Then:

1. $(X/Y)^* \cong Y^\perp$ topologically via $f \mapsto f \circ Q$, where $Q : X \rightarrow X/Y$ is the quotient map.
2. $X^*/Y^\perp \cong Y^*$ topologically via $f + Y^\perp \mapsto f|Y$.

Moreover, if $X$ and $Y$ are normed then the above topological isomorphisms are isometric.

**Closed Range Theorem.** Let $X$ and $Y$ be Banach spaces and $T \in B(X,Y)$. The following are equivalent:

1. $T$ has closed range;
2. $T^*$ has closed range;

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3. $T^*$ has weak*-closed range.

The double dual of a normed space $X$ is the dual $X^{**}$ of $X^*$, where $X^*$ is given the norm topology.

**Proposition.** Let $X$ be a normed space. The map $\phi: X \to X^{**}$ defined by

$$\phi(x)(g) = g(x) \quad \text{for } x \in X, g \in X^*$$

is an isometry onto a weak*-dense subspace.

A Banach space $X$ is reflexive if the above map $\phi: X \to X^{**}$ is onto.

**Theorem.** A Banach space is reflexive if and only if its closed unit ball is weakly compact.