If a topological vector space has a compatible invariant metric, then the topological vector space and the metric space do not necessarily have the same bounded subsets. In particular, the metric can be chosen to be bounded, but a nonzero topological vector space is never bounded.

**Proposition.** Every Cauchy sequence in a topological vector space is bounded.

**Proposition.** A set $A$ in a topological vector space is bounded if and only if $c_n x_n \to 0$ whenever $x_n \in A$ and $c_n \to 0$ in $\mathbb{F}$.

A linear map between topological vector spaces is *bounded* if the image of every bounded set is bounded.

**Proposition.** Let $X$ and $Y$ be topological vector spaces and $T : X \to Y$ linear. Consider the following properties:

1. $T$ is continuous;
2. $T$ is bounded;
3. $\{Tx_n\}$ is bounded whenever $x_n \to 0$.

Then $1 \implies 2 \implies 3$.

Moreover, if $X$ is metrizable then all of the above properties are equivalent to:

4. $Tx_n \to 0$ whenever $x_n \to 0$.

If $X$ and $Y$ are normed spaces, then $L(X, Y)$ is usually written $B(X, Y)$, in view of the above proposition. The *operator norm* of $T \in B(X, Y)$ is

$$\|T\| := \sup\{\|Tx\| : \|x\| \leq 1\}.$$

**Proposition.** If $X$ and $Y$ are normed spaces, then $B(X, Y)$ is a normed space with the operator norm, and is complete if $Y$ is. Moreover $\|T\|$ is the minimum $k \geq 0$ satisfying

$$\|Tx\| \leq k\|x\|$$

for all $x \in X$.

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