Let $C(\mathbb{R})$ have the projective topology generated by the seminorms
\[ p_k(f) := \sup\{|f(x)| : |x| \leq k\} \quad \text{for } k \in \mathbb{N}. \]

1. Prove that $f_n \to f$ in $C(\mathbb{R})$ if and only if, for every compact set $A \subset \mathbb{R}$ we have $f_n \to f$ uniformly on $A$.

2. Prove that for every $f \in C(\mathbb{R})$ there exists a sequence $\{f_n\}$ in $C^0_0(\mathbb{R})$ (the set of continuous functions vanishing at infinity) which converges to $f$ in $C(\mathbb{R})$. 

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