Exercise 1. Prove that if $A$ is bounded in a topological vector space, then for every neighborhood $V$ of 0 there exists $c > 0$ such that $tV \supset A$ for all $t > c$.

Exercise 2. Prove that, in a topological vector space, a set $A$ is bounded if and only if every countable subset of $A$ is bounded.

Exercise 3. Prove that if $A$ and $B$ are convex, then so is $A + B$.

Exercise 4. Prove that if $A$ and $B$ are balanced, then so is $A + B$.

Exercise 5. Prove that if $A$ and $B$ are bounded, then so is $A + B$.

Exercise 6. Prove that if $A$ and $B$ are compact, then so is $A + B$.

Exercise 7. Prove that if $A$ is compact and $B$ is closed, then $A + B$ is closed.

Exercise 8. For $r > 0$ put $V_r = \{ f \in \mathbb{R}^\mathbb{R} : |f(x)| < r \text{ for all } x \in \mathbb{R} \}$, and give $\mathbb{R}^\mathbb{R}$ the topology generated by the sets $f + V_r$ for $f \in \mathbb{R}^\mathbb{R}$ and $r > 0$. Prove that addition is continuous but scalar multiplication is not.

Exercise 9. Prove:
   
   (a) $\mathbb{R}^\mathbb{N}$ is metrizable;
   (b) $\mathbb{R}^\mathbb{N}$ is not locally bounded;
   (c) $\mathbb{R}^\mathbb{R}$ is not metrizable.

Exercise 10. Let $X$ and $Y$ be topological vector spaces and let $T : X \to Y$ be linear. Prove that if dim $Y < \infty$ and ker $T$ closed, then $T$ is continuous.

Exercise 11. Let $V$ be a convex neighborhood of 0 in $L^p := L^p[0, 1]$, where $0 < p < 1$. Prove that $V = L^p$.

Hint: Let $f \in L^p$. Choose $r > 0$ such that $B_r(0) \subset V$. Why does there exist $n \in \mathbb{N}$ such that $n^{p-1}d(f, 0) < r$? Why do there exist

\[ 0 = x_0 < x_1 < \cdots < x_n = 1 \]

Date: September 1, 2005.
such that
$$\int_{x_{i-1}}^{x_i} |f|^p = \frac{d(f, 0)}{n} \quad \text{for } i = 1, \ldots, n?$$

Put $f_i = n f(x_{i-1}, x_i)$ for $i = 1, \ldots, n$. Why is each $f_i \in V$ and
$$f = \frac{1}{n} \sum_{i=1}^n f_i?$$

**Exercise 12.** Let $p$ be a seminorm on a vector space $X$. Put $N = p^{-1}(0)$. Then $N$ is a subspace of $X$, and the function $\tilde{p}: X/N \to \mathbb{R}$ defined by
$$\tilde{p}(x + N) = p(x)$$
is a norm.

**Exercise 13.** Prove that the closed unit ball of $c_0$ is not weakly compact.
Hint: $(c_0)^* = \ell^1$.

**Exercise 14.** Prove that $\ell^1$ is not reflexive.

**Exercise 15.** Prove that the closed unit ball in $L^1[0, 1]$ has no extreme points. What can you conclude about $L^1[0, 1]$ being a dual space?

**Exercise 16.** Let $X$, $Y$, $Z$, and $W$ be Banach spaces. Prove that
$$K(X, Y)B(Z, X) \subset K(Z, Y) \quad \text{and} \quad B(Y, W)K(X, Y) \subset K(X, W).$$

**Exercise 17.** Prove that a subspace of $L^2[0, 1] \cap C[0, 1]$ which is closed in $L^2[0, 1]$ must be finite-dimensional.

Hint: Show that the subspace is also closed in $C[0, 1]$.

**Exercise 18.** Prove that in a locally convex space, a subset is bounded if and only if its polar is absorbing.

**Exercise 19.** Prove that a subspace of a locally convex space is dense if and only if its annihilator is $\{0\}$.

**Exercise 20.** Let $X$ and $Y$ be locally convex spaces, and let $T \in L(X, Y)$. Prove that $T$ has dense range if and only if $T^*$ is 1-1, and $T$ is 1-1 if and only if $T^*$ has weak*-dense range.

**Exercise 21.** Let $X$ be an infinite-dimensional locally convex space. Prove that every weak neighborhood of 0 contains a nontrivial subspace.

**Exercise 22.** Let $X$ be an infinite-dimensional Banach space. Prove that the unit sphere \( \{x : \|x\| = 1\} \) in $X$ is dense in the closed unit ball.

**Exercise 23.** Let $X$ be an infinite-dimensional Fréchet space, and let the dual $X^*$ have the weak*-topology. Prove:
Exercise 24. Prove that if $U \subset \mathbb{R}^n$ is open then $C(U)$, with the topology of uniform convergence on compact sets, is not Heine-Borel.

Exercise 25. Prove that if $U \subset \mathbb{C}$ is open then $H(U)$ is Heine-Borel.

Exercise 26. Let $X$ be the set of continuously differentiable functions on $[0, 1]$, regarded as a subspace of $C[0, 1]$. Define a linear map $D: X \to C[0, 1]$ by $Df = f'$. Prove that $D$ is closed and unbounded.

Exercise 27. Let $X$ be the set of continuously differentiable functions on $[0, 1]$, with the norm $\|f\| := \|f\|_u + \|f'\|_u$, where $\| \cdot \|_u$ denotes the uniform norm for bounded functions on $[0, 1]$. Prove that $X$ is a Banach space.

Exercise 28. With $X$ as in the preceding exercise, prove that the linear map $D: X \to C[0, 1]$ defined by $Df = f'$ is bounded, and find $\|D\|$. 

Exercise 29. With $X$ as in the preceding exercise, prove that the inclusion map $X \hookrightarrow C[0, 1]$ is compact.

Exercise 30. Prove that if $0 < p < q \leq \infty$ then $L^q[0, 1]$ is meager in $L^p[0, 1]$.

Exercise 31. Prove that every metric space $X$ can be isometrically embedded in the Banach space $C_b(X)$.

Hint: For fixed $t \in X$, consider the functions $f_y(x) = d(x, y) - d(x, t)$ for $y \in X$.

Exercise 32. Prove that every normed space is isometrically isomorphic to a subspace of $C(X)$ for some compact Hausdorff space $X$.

Hint: Alaoglu.

Exercise 33. (Completion) Let $X$ be a normed space, and let $\phi: X \to X^{**}$ be the canonical embedding. Let $Y$ be the norm closure of $\phi(X)$ in $X^{**}$.

(a) Prove that if $Z$ is a Banach space and $T \in B(X, Z)$ then there exists $S \in B(Y, Z)$ such that $S\phi = T$.

(b) Suppose that $\pi$ is an isometric linear map from $X$ onto a dense subspace of a Banach space $W$. Prove that there is an isometric isomorphism $R: Y \to W$ such that $R\phi = \pi$.

Exercise 34. Prove that a sequence is weakly convergent in $C[0, 1]$ if and only if it is bounded and pointwise convergent.
Exercise 35. Let \( \{e_n\} \) be an orthonormal basis for a separable Hilbert space \( H \), and let \( \{c_n\} \) be a bounded sequence in \( \mathbb{F} \).

(a) Prove that there is a unique \( T \in L(H) \) such that

\[
Te_n = c_n e_n \quad \text{for all } n \in \mathbb{N}.
\]

(b) Prove that the operator \( T \) from part (a) is compact if and only if \( c_n \to 0 \).

Exercise 36. Let \( K \in C([0,1]^2) \), and define a linear operator \( T \) on \( L^2[0,1] \) by

\[
Tf(x) = \int_0^1 K(x, y) f(y) \, dy \quad \text{for } f \in L^2[0,1].
\]

Prove that \( T \) is compact.

Exercise 37. Prove that if \( \{p_n\} \) is a pointwise-bounded sequence of seminorms on a vector space, then \( \limsup p_n \) is a seminorm.

Exercise 38. (Banach Limits) Prove that there exists \( \Lambda \in (\ell^\infty)^* \) such that

- \( \Lambda x = \lim x_n \) if \( x \) converges, and
- \( \Lambda S = \Lambda \), where \( S \in L(\ell^\infty) \) is defined by \( (Sx)_n = x_{n+1} \).

Hint: Apply the Hahn-Banach Theorem to the seminorm

\[
p(x) = \limsup_{n \to \infty} \left| \frac{1}{n} \sum_{k=1}^n x_k \right|
\]

and a suitable linear functional on the subspace of convergent sequences.

Exercise 39. With the notation and terminology of the preceding exercise, prove that no Banach Limit is in \( \ell^1 \).

Exercise 40. Prove that a normed space \( X \) is complete if and only if for every series series \( \sum_{n=1}^\infty x_n \) in \( X \), if \( \sum_{n=1}^\infty \|x_n\| < \infty \) (i.e., the series is absolutely convergent) then \( \sum_{n=1}^\infty x_n \) converges.

Hint: for the converse direction, given a Cauchy sequence \( (y_k) \), choose a subsequence \( (z_j) \) such that \( \sum_{j=1}^\infty \|z_{j+1} - z_j\| < \infty \).

Exercise 41. (Riemann-Lebesgue Lemma) Prove that \( z^n \to 0 \) weak* in \( L^\infty(\mathbb{T}) \).

Exercise 42. Use the Uniform Boundedness Principle to prove that there exists \( f \in L^2(\mathbb{T}) \) whose Fourier series

\[
\sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx}
\]

diverges at 0.
Hint: the $n$th partial sum evaluated at $x = 0$ is

$$\phi_n(f) = \sum_{k=-n}^{n} \hat{f}(k) = \int_0^{2\pi} f D_n,$$

where $D_n$ is the Dirichlet kernel

$$D_n(x) = \sum_{k=-n}^{n} e^{ikx}.$$

It follows from Hilbert space theory that $\phi_n \in L^2(\mathbb{T})^*$ with $\|\phi_n\| = \|D_n\|_2$. Show that $\|D_n\|_2 \to \infty$ as $n \to \infty$, and show how the UBP applies.

**Exercise 43.** With the notation and terminology of the preceding exercise, prove that there exists $f \in C(\mathbb{T})$ whose Fourier series diverges at 0.

Hint:

$$D_n(x) = \frac{\sin(n + 1/2)x}{\sin x/2},$$

and it follows from integration theory that $\phi_n \in C(\mathbb{T})^*$ with

$$\|\phi_n\| = \|D_n\|_1 = \int_0^{2\pi} |D_n|.$$

Show that $\|D_n\|_1 \to \infty$ (this is a little challenging).

**Exercise 44.** Prove that, in a normed space, if $x_n \to x$ weakly, then $\|x\| \leq \lim inf \|x_n\|$.

Hint: Hahn-Banach.

**Exercise 45.** Let $A$ be a subset of a Banach space $X$ whose linear span is dense, let $f \in X^*$, and let $\{f_n\}$ be a bounded sequence in $X^*$. Prove that if $f_n(x) \to f(x)$ for all $x \in A$, then $f_n \to f$ weak*.

**Exercise 46.** Prove that the delta functions $\delta_n$ converge weakly in $\ell^p$ for $1 < p \leq \infty$ but not for $p = 1$.

**Exercise 47.** Prove that no infinite-dimensional $F$-space has a countable Hamel basis.

Hint: consider finite-dimensional subspaces.

**Exercise 48.** Prove that $C[0,1]$ is weak*-dense in $L^\infty[0,1]$.

**Exercise 49.** For each $x \in [0,1]$ let $\delta_x$ be the point mass at $x$, and let $K = \{\delta_x : x \in [0,1]\}$. Let $P$ be the set of probability measures on $[0,1]$, i.e., the regular Borel (positive) measures $\mu$ such that $\mu([0,1]) = 1$. Prove:

(a) $P$ is convex and weak*-compact in $M(X)$.
(b) $K$ is weak*-compact in $M(X)$. 
(c) $K$ is the set of extreme points of $P$.

Hint: use Hahn-Banach to show that $P$ is contained in the weak*-convex closure of $K$.

**Exercise 50.** Let $X$ be a Banach space. Prove that if $X^*$ is separable, then so is $X$.

**Exercise 51.** Let $X$ be a Banach space, let $B$ be the unit ball of $X$, and let $\phi: X \to X^{**}$ be the canonical embedding. Prove that $\phi(B)$ is weak*-dense in $X^{**}$.

**Exercise 52.** Let $X$ be a topological vector space, and let $f \in X^*$. Describe the adjoint map $f^*$.

**Exercise 53.** Let $X$ be a Banach space.

(a) Prove that in $B(X)$, the limit of a convergent sequence of finite-rank operators is compact.

(b) Prove that if $X$ is a Hilbert space, then every compact operator is the limit of a sequence of finite-rank operators.

**Exercise 54.** Let $X$ be a Banach space and $T \in B(X)$.

(a) Prove that if $T$ is compact, then $\|Tx_n\| \to 0$ whenever $x_n \to 0$ weakly.

(b) Prove that if $X$ is a Hilbert space, and if $T$ has the property that $\|Tx_n\| \to 0$ whenever $x_n \to 0$ weakly, then $T$ is compact.