TYCHONOFF

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Theorem 1 (Tychonoff). Every product of compact spaces is compact.

Proof. Let $X_i$ be compact for $i \in I$. Put $X = \prod_{i \in I} X_i$, and for each $i \in I$ let $\pi_i : X \to X_i$ be the coordinate projection.

It follows straight from the definitions that it suffices to show that if $\mathcal{D}$ is a family of closed subsets of $X$ with the Finite Intersection Property (that is, every finite subfamily of $\mathcal{D}$ has nonempty intersection), then $\bigcap \mathcal{D} \neq \emptyset$. Use Zorn’s Lemma to find a maximal family $\mathcal{E} \supset \mathcal{D}$ of (not necessarily closed) subsets of $X$ with the Finite Intersection Property.

For each $i \in I$, the family $\{\pi_i(A) \mid A \in \mathcal{E}\}$ of subsets of $X_i$ has the Finite Intersection Property, so we can choose $x_i \in \bigcap_{A \in \mathcal{E}} \pi_i(A)$ since $X_i$ is compact. Put $x = (x_i) \in X$. It suffices to show that $x \in \overline{A}$ for all $A \in \mathcal{E}$.

Let $U \subset X$ be open with $x \in U$. Without loss of generality $U = \bigcap_{i \in J} \pi_i^{-1}(V_i)$ for some finite $J \subset I$ and open $V_i \subset X_i$ for each $i \in J$. For each $i \in J$ we have $x_i \in V_i$, so for all $A \in \mathcal{E}$ we have $V_i \cap \pi_i(A) \neq \emptyset$, hence $\pi_i^{-1}(V_i) \cap A \neq \emptyset$. It follows from maximality that $\pi_i^{-1}(V_i) \in \mathcal{E}$ for all $i \in J$. Thus for all $A \in \mathcal{E}$ we have $U \cap A \neq \emptyset$ by the Finite Intersection Property. □