Throughout this discussion $X$ will be a locally compact Hausdorff space, with one-point compactification $X^*$. We assume the basic properties of $X^*$.

**Proposition 0.1.** If $K \subset U \subset X$ with $K$ compact and $U$ open, there exist an open set $V$ and a compact set $L$ such that

$$K \subset V \subset L \subset U.$$  

**Proof.** $K$ is closed in $X^*$ and $U$ is open in $X^*$. Since $X^*$ is compact Hausdorff, hence normal, there exist $V$ and $L$ such that $K \subset V \subset L \subset U$, $V$ is open $X^*$, and $L$ is (closed in $X^*$, hence) compact. Then $V$ is also open in $X$. □

**Theorem 0.2** (Urysohn). If $K \subset U \subset X$ with $K$ compact and $U$ open, there exists $f \in C_c(X, [0, 1])$ such that $f \equiv 1$ on $K$ and $\text{supp} \ f \subset U$.

**Proof.** Apply the above proposition to get $V$ and $L$. Since $X^*$ is normal there exists $g \in C(X^*, [0, 1])$ which is identically 1 on $K$ and 0 outside $V$. Then $f := g|A \in C(X, [0, 1])$ is identically 1 on $K$ and vanishes outside the compact subset $L$ of $U$. The result follows. □

**Theorem 0.3** (Tietze). If $K \subset X$ is compact and $f \in C(K)$, then there exists $g \in C_c(X)$ extending $f$.

**Proof.** Apply Tietze to $X^*$ to get $\phi \in C(X^*)$ extending $f$, and let $\psi = \phi|X \in C(X)$. Apply the above version of Urysohn to choose $\theta \in C_c(X)$ which is identically 1 on $K$. Then $g := \psi \theta \in C_c(X)$ extends $f$. □

Now we assume the definition of $C_0(X)$. It follows straight from the definitions that

$$C_0(X) = \{f|X \mid f \in C(X^*), f(\infty) = 0\},$$

and the (uniform) norm of $f \in C(X^*)$ coincides with the norm of $f|X$ in $C_b(X)$.

**Theorem 0.4.** $C_0(X)$ is the closure of $C_c(X)$ in $C_b(X)$.
Proof. The characterization of $C_0(X)$ mentioned immediately before the current theorem shows that $C_0(X)$ is closed in $C_b(X)$. Let $f \in C_0(X)$ and $\epsilon > 0$. Choose a compact set \( K \subset X \) such that \(|f| < \epsilon\) off \( K \). Then choose \( g \in C_c(X, [0, 1]) \) such that \( g \equiv 1 \) on \( K \). Then \( fg \in C_c(X) \). We have \( fg = f \) on \( K \), and for \( x \notin K \) we have

\[ |f(x) - f(x)g(x)| = |f(x)| (1 - g(x)) \leq |f(x)| < \epsilon. \]

Thus \( \|f - fg\| < \epsilon \). \( \square \)