Final Exam

Time limit: 1 hour, 50 minutes. No notes, books, calculators, etc. Write your solutions on the blank pages provided — nothing on the test page(s) will be graded. Write on only one side of each page and leave reasonable margins. Print your name clearly in the upper right corner of each page, including the test page(s). Turn in your exam with the test page(s) on top. Write your final solutions clearly and in logical order — do not turn in scratch pages. You must follow all explicit instructions and show all your reasoning. This exam has 5 problems, and all problems have equal credit.

1. In each part, answer True or False — no reasons needed.
   (a) The step functions are dense in $L^1[0,1]$.  
   (b) $L^1[0,1] \subset L^2[0,1]$. 
   (c) Counting measure is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$. 
   (d) If $f$ is integrable on $[0,1]$ and $F(x) = \int_0^x f$, then $F' = f$ a.e. 
   (e) In the Banach space $C[0,1]$, every closed and bounded set is compact. 
   (f) If $\nu$ is a complex measure on $\mathcal{M}$, then $\sup\{|\nu(A)| : A \in \mathcal{M}\} < \infty$.

2. Show that there is a linear functional $\phi$ on $L^\infty[0,1]$ such that 
   $$|\phi(f)| \leq \|f\|_\infty \quad \text{for all } f \in L^\infty[0,1]$$ 
   and 
   $$\phi(f) = f(0) \quad \text{for all } f \in C[0,1].$$

3. With $\phi$ as in Problem 2 (and assuming the result of Problem 2 is true), prove that there does not exist $g \in L^1[0,1]$ such that 
   $$\phi(f) = \int_0^1 fg \quad \text{for all } f \in L^\infty[0,1].$$

4. Find, with full justification:
   $$\lim_{n \to \infty} \int_0^1 \frac{1 + nx^2}{(1 + x^2)^n} \, dx$$

5. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be sigma-finite measure spaces, and let $f \in L^2(X \times Y, \mu \times \nu)$. Prove that, if we define $f_x(y) = f(x,y)$, then $f_x \in L^2(\nu)$ for $\mu$-almost all $x$, and 
   $$\|f\|_2^2 = \int_X \|f_x\|_2^2 \, d\mu(x).$$