Integrating nonnegative functions

**Notation.** Thoughout this section \((X, \mathcal{M}, \mu)\) will denote a \(\sigma\)-finite measure space, and \(S^+\) will denote the set of nonnegative simple functions on \(X\).

**Definition 1.** For \(\phi \in S^+\) we define \(\int \phi \, d\mu = \sum_1^n c_i \mu(A_i)\) if \(\phi = \sum_1^n c_i \chi_{A_i}\) with \(c_1, \ldots, c_n \geq 0\) and \(A_1, \ldots, A_n\) disjoint. If \(A \in \mathcal{M}\) we define \(\int_A \phi \, d\mu = \int \phi \chi_A \, d\mu\). When the measure \(\mu\) is understood we write \(\int \phi\), and when we want to emphasize the space we write \(\int_X \phi\). Thus, with the above notation, \(\int_A \phi = \sum_1^n c_i \mu(A_i \cap A)\).

Note that each product \(c_i m(A_i)\) is computed in \([0, \infty]\); although \(c_i\) is a nonnegative real number, \(m(A_i)\) could be \(\infty\), in which case \(c_i m(A_i)\) is \(\infty\) if \(c_i > 0\) and 0 if \(c_i = 0\).

**Proposition 2.** For all \(\phi, \psi \in S^+\) we have:

1. \(\int \phi\) is well-defined;
2. \(\int \phi \leq \int \psi\) if \(\phi \leq \psi\);
3. \(\int c\phi = c \int \phi\) if \(c \in [0, \infty)\);
4. \(\int (\phi + \psi) = \int \phi + \int \psi\).

**Proof.** Let \(\phi = \sum_1^n c_i \chi_{A_i}\), with \(c_1, \ldots, c_n \geq 0\) and \(A_1, \ldots, A_n \in \mathcal{M}\) disjoint, and similarly for \(\psi = \sum_1^k d_j \chi_{B_j}\). If necessary, add terms with coefficients equal to 0 so that without loss of generality \(\bigcup_1^n A_i = \bigcup_1^k B_j\). Next, subdivide the \(A_i\)’s and \(B_j\)’s so that without loss of generality \(n = k\) and \(A_i = B_i\) for all \(i\). A moment’s reflection reveals that subdividing the \(A\)’s doesn’t change \(\sum c_i m(A_i)\), and similarly for the \(B\)’s. So, we now have \(\phi = \sum_1^n c_i \chi_{A_i}\) and \(\psi = \sum_1^k d_j \chi_{B_j}\). (1)–(4) are now obvious.

**Proposition 3.** If \(\phi \in S^+\) then the map \(A \mapsto \int_A \phi\) is a measure on \(\mathcal{M}\).

**Proof.** Let \(\phi = \sum_1^n c_i \chi_{A_i}\) as above. First of all, \(\int \phi \geq 0\) by construction. Next,

\[\int \phi = \int \phi \chi_\emptyset = \int 0 = 0.\]

Finally, if \(C_1, C_2, \ldots \in \mathcal{M}\) are disjoint, then

\[\int \bigcup_1^\infty n \phi = \sum_1^n c_i \mu(A_i \cap \bigcup_1^\infty C_n) = \sum_1^n c_i \mu(\bigcup_1^\infty (A_i \cap C_n)) = \sum_1^n c_i \sum_1^\infty \mu(A_i \cap C_n) = \sum_1^n c_i \mu(A_i \cap C_n) = \int \phi.\]
**Definition 4.** \(L^+ = L^+(X, \mathcal{M})\) denotes the set of nonnegative measurable functions on \(X\). For \(f \in L^+\) we define \(\int f \, d\mu = \sup \{ \int \phi \, d\mu : \phi \in S^+, \phi \leq f \}\). If \(A \in \mathcal{M}\) we define \(\int_A f \, d\mu = \int f \chi_A \, d\mu\). When the measure \(\mu\) is understood we write \(\int f\), and when we want to emphasize the space we write \(\int_X f\).

**Observation 5.** For \(\phi \in S^+\) the two definitions of \(\int \phi\) are consistent, by monotonicity.

**Proposition 6.** For all \(f, g \in L^+\),

1. \(\int f \leq \int g\) if \(f \leq g\);
2. \(\int cf = c \int f\) if \(c \in [0, \infty)\);

**Proof.** (1) If \(\phi\) is simple with \(0 \leq \phi \leq f\), then \(\phi \leq g\), so \(\int \phi \leq \int g\). Taking the sup over \(\phi\), we get \(\int f \leq \int g\).

(2) Put

\[
\mathcal{F} = \{ \phi \in S^+ : \phi \leq f \} \quad \mathcal{F}' = \{ \psi \in S^+ : \psi \leq cf \}.
\]

Then \(\mathcal{F}' = \{ c\phi : \phi \in \mathcal{F} \}\), so

\[
c \int f = c \sup_{\phi \in \mathcal{F}} \int \phi = \sup_{\phi \in \mathcal{F}} c \int \phi = \sup_{\phi \in \mathcal{F}} \int c\phi = \sup_{\psi \in \mathcal{F}'} \psi = \int cf. \quad \square
\]

**Theorem 7** (Monotone Convergence Theorem). If \((f_n)\) is a sequence in \(L^+(X)\) and \(f_n \uparrow f\) pointwise, then

\[
\int f_n \to \int f.
\]

**Proof.** Let \(\phi \in S^+\) with \(\phi \leq f\). Let \(0 < a < 1\), and put \(A_n = [f_n \geq a\phi]\). Then \(f_n \geq a\phi \chi_{A_n}\), so

\[
\int f_n \geq \int A_n a\phi = a \int A_n \phi.
\]

Since \(A_1 \subset A_2 \subset \cdots\) and \(\bigcup_n A_n = X\), we have

\[
\int A_n \phi \to \int \phi
\]

because \(B \mapsto \int_B \phi\) is a measure. Since \(f_1 \leq f_2 \leq \cdots\), the sequence \((\int f_n)\) is increasing, so \(\lim \int f_n\) exists in \(\mathbb{R}\), and

\[
\lim \int f_n \geq a \int \phi.
\]

Letting \(a \uparrow 1\), we get

\[
\lim \int f_n \geq \int \phi.
\]

Taking the sup over \(\phi\), we find

\[
\lim \int f_n \geq \int f.
\]

On the other hand, \(f_n \leq f\), so \(\int f_n \leq \int f\), hence

\[
\lim \int f_n \leq \int f.
\]
Therefore we must have \( \lim \int f_n = \int f \). \hfill \Box

**Corollary 8.** If \( f, g \in L^+ \) then \( \int (f + g) = \int f + \int g \).

**Proof.** Choose sequences \((\phi_n), (\psi_n)\) in \( S^+ \) such that \( \phi_n \uparrow f \) and \( \psi_n \uparrow g \). Then \((\phi_n + \psi_n) \uparrow (f + g)\), so by the Monotone Convergence Theorem
\[
\int (f + g) = \lim \int (\phi_n + \psi_n) = \lim \left( \int \phi_n + \int \psi_n \right) \\
= \lim \int \phi_n + \lim \int \psi_n \\
= \int f + \int g. \hfill \Box
\]

**Corollary 9** (Series Version of MCT). If \( \sum f_n \) is a series in \( L^+ (X) \), then \( \int \sum f_n = \sum \int f_n \).

**Proof.** The partial sums \( \sum_{1}^{\infty} f_n \) are nonnegative and increase to \( \sum_{1}^{\infty} f \), so by the Monotone Convergence Theorem
\[
\int \sum_{1}^{\infty} f_n = \lim \int \sum_{1}^{k} f_n \\
= \lim \sum_{1}^{k} \int f_n \quad \text{(by Corollary 8 and induction)} \\
= \sum_{1}^{\infty} \int f_n. \hfill \Box
\]

When we apply the above corollary, we’ll just say “by the Monotone Convergence Theorem”.

**Corollary 10.** If \( f \in L^+ \) then \( A \mapsto \int_A f \) is a measure on \( \mathcal{M} \).

**Proof.** \( \int_A f \geq 0 \) by construction, and
\[
\int_{\emptyset} f = \int f \chi_{\emptyset} = \int 0 = 0.
\]
If \( A_1, A_2, \ldots \in \mathcal{M} \) are disjoint, then by the Monotone Convergence Theorem
\[
\int_{\bigcup A_n} f = \int f \chi_{\bigcup A_n} = \int f \sum \chi_{A_n} \\
= \sum \int f \chi_{A_n} = \sum \int_{A_n} f. \hfill \Box
\]

**Proposition 11.** If \( f \in L^+ \), then \( \int f = 0 \) if and only if \( f = 0 \) a.e..

**Proof.** First assume \( \int f = 0 \). For each \( n \in \mathbb{N} \) put \( A_n = [f > 1/n] \). Then \( [f \neq 0] = \bigcup_n A_n \), so it suffices to observe that \( \mu(A_n) = 0 \) for each \( n \), which follows from
\[
0 = \int f \geq \int_{A_n} f \geq \frac{\mu(A_n)}{n}.
\]
Conversely, assume that $f = 0$ a.e. Let $\phi \in S^+$ with $\phi \leq f$, and write $\phi = \sum_1^n c_i \chi_{A_i}$ with $c_i \geq 0$. Since $f = 0$ a.e., we have $\mu(A_i) = 0$ for each $i$ with $c_i > 0$. Thus

$$\int \phi = \sum_1^n c_i \mu(A_i) = 0.$$  

Taking the sup over $\phi$ gives $\int f = 0$. □

**Corollary 12.** If $f \in L^+$ and $\mu(A) = 0$ then $\int_A f = 0$ and $\int_{A^c} f = \int f$.

**Proof.** We have $f \chi_A = 0$ a.e., so $\int_A f = 0$, and therefore

$$\int f = \int_A f + \int_{A^c} f = \int f.$$  □

**Corollary 13.** For all $f, g \in L^+$, if $f = g$ a.e. then $\int f = \int g$.

**Proof.** If $\mu(A^c) = 0$ and $f = g$ on $A$, then

$$\int f = \int_A f = \int_A g = \int g.$$  □

**Corollary 14.** In the Monotone Convergence Theorem, $f_n \uparrow f$ a.e. is ok.

**Proof.** If $f_n \uparrow f$ on $A$ and $\mu(A^c) = 0$, then by the Monotone Convergence Theorem

$$\int f_n = \int_A f_n \to \int_A f = \int f.$$  □

**Proposition 15.** For all $f \in L^+$, if $\int f < \infty$ then $f < \infty$ a.e.

**Proof.** Put $A = [f = \infty]$. Suppose $\mu(A) > 0$. Then for all $n \in \mathbb{N}$ we have $f \geq n \chi_A$, so $\int f \geq n \mu(A)$. Thus we must have $\int f = \infty$. Contrapositively, if $\int f < \infty$ then $\mu(A) = 0$, so $f < \infty$ a.e. □

**Theorem 16** (Fatou’s Lemma). If $(f_n)$ is a sequence in $L^+(X)$, then

$$\int \liminf f_n \leq \liminf \int f_n.$$  

In particular, if $f_n \to f$ a.e., then $\int f \leq \liminf \int f_n$.

**Proof.** Put $g_k = \inf_{n \geq k} f_n$. Then $0 \leq g_k \uparrow \liminf f_n$, and $g_k \leq f_k$ for all $k$, so by the Monotone Convergence Theorem

$$\int \liminf f_n = \lim \int g_k \leq \liminf \int f_k.$$  □