1. Let $f \in L^+(\mathbb{R})$ and $c \in \mathbb{R}$. Prove that

$$\int_{\mathbb{R}} f(x + c) \, dx = \int_{\mathbb{R}} f.$$

Hint: first do it for simple functions, then use the Monotone Convergence Theorem. Note that integrals over $\mathbb{R}$ mean (by default) with respect to Lebesgue measure.

2. (Borel-Cantelli Lemma) Let $\{A_n\}$ be a sequence of measurable sets in a measure space $(X, \mu)$, and suppose that $\sum_{1}^{\infty} \mu(A_n) < \infty$. Prove that almost every point is in only finitely many $A_n$’s. That is, show that there exists $B \subset X$ such that $m(B) = 0$ and for all $x \notin B$, the set $\{n \in \mathbb{N} : x \in A_n\}$ is finite. This exercise occurred already in the section on measures, but now do it by applying the Monotone Convergence Theorem to the functions $\chi_{A_n}$. 

Integrating nonnegative functions — Exercises