Implicit functions — Exercises

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $C^1$ and $(a, b) \in \mathbb{R}^2$. Assume that $f'(a, b) \neq 0$. Use the Implicit Function Theorem to prove that $f$ is not 1-1.

2. Let $E \subset \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^m$, and let $f : E \to \mathbb{R}^m$ be $C^1$. Let $(a, b) \in E$, and write $f'(a, b) = (A \ B)$ where $B$ is $m \times m$. Prove that if $B$ is invertible then $f(a, b)$ is in the interior of the range of $f$.
   
   Hint: use the proof of the Implicit Function Theorem.

3. Let $E \subset \mathbb{R}^n$ and $f : E \to \mathbb{R}^m$. Assume that $f$ is $C^1$ and $f'(x)$ is onto for all $x \in E$. Prove that $f(E)$ is open.

4. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ be $C^1$. Prove that the set
   
   $\{ x \in U : f'(x) \text{ is onto} \}$
   
   is open.