Linear maps — Exercises

1. Fix $x \in \mathbb{R}^n$, and define $S \in L(\mathbb{R}, \mathbb{R}^n)$ and $T \in L(\mathbb{R}^n, \mathbb{R})$ by $St = tx$ and $Ty = x \cdot y$. Prove that
$$||S|| = ||T|| = ||x||.$$

2. Let $T \in L(X, Y)$ be onto. Prove that if $U \subset X$ is open then so is $T(U)$. Hint: Let $W = \ker T$. Then there is a subspace $Z$ of $X$ such that
$$\dim Z = \dim Y, \quad X = W + Z, \quad \text{and} \quad W \cap Z = \{0\}.$$

3. (a) Let $Z$ be a subspace of $X$ with the same dimension as $Y$. Prove that the set
$$\{T \in L(X, Y) : T(Z) = Y\}$$
is open in $L(X, Y)$.

(b) Let $F \subset X$ be linearly independent and have the same number of elements as the dimension of $Y$. Prove that the set
$$\{T \in L(X, Y) : T(F) \text{ is linearly independent}\}$$is open in $L(X, Y)$.

(c) Prove that the set
$$\{T \in L(X, Y) : T \text{ is onto}\}$$is open in $L(X, Y)$.

4. Let $m < n$, and let
$$1 \leq j_1 < j_2 < \cdots < j_m \leq n.$$
Prove that
$$\{A \in M_{m \times n} : A_{j_1}, \ldots, A_{j_m} \text{ are linearly independent}\}$$is open in $M_{m \times n}$, where $A_j$ denotes the $j$th column of $A$.

5. Let $A \in M_n$. Prove that
$$|x^t Ax| \leq ||A|| ||x||^2 \quad \text{for all } x \in \mathbb{R}^n,$$where $x^t$ denotes the transpose of $x$.

6. A symmetric $n \times n$ matrix $A$ is called positive definite if $x^t Ax > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

(a) Prove that if $x^t Ax > 0$ whenever $||x|| = 1$, then $A$ is positive definite.
(b) Formulate an appropriate definition of \textit{negative definite}, and state and prove a result analogous to part (a) if $A$ is negative definite. Hint: to make it easier, show that $A$ is negative definite if and only if $-A$ is positive definite (but \textbf{this should not be your definition of negative definite!}).

7. Let $A \in M_n$.
   
   (a) Suppose $A$ is positive definite. Prove that there exists $\varepsilon > 0$ such that for every symmetric $B \in M_n$, if $\|B - A\| < \varepsilon$ then $B$ is positive definite.
   
   Hint: show that $\inf \{ x^t A x : \|x\| = 1 \} > 0$.
   
   (b) Formulate and prove a result analogous to part (a) if $A$ is negative definite.

8. Let $A \in M_n$ and $x \in \mathbb{R}^n$.
   
   (a) Suppose $x^t A x > 0$. Prove that there exists $\varepsilon > 0$ such that for all $B \in M_n$, if $\|B - A\| < \varepsilon$ then $x^t B x > 0$.
   
   (b) State and prove a result analogous to part (a) if $x^t A x < 0$. 