Theorem. Let $T : X \to Y$ be linear, with $\dim X < \infty$. Then:

(1) $T$ is bounded;

(2) If $T$ is an isomorphism then $T^{-1}$ is bounded.

Proof. We first prove both (1) and (2) for the case $X = \mathbb{R}^n$. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$. Then for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we have

$$\|Tx\| = \left\| T \sum_{i=1}^{n} x_i e_i \right\| = \left\| \sum_{i=1}^{n} x_i Te_i \right\| \leq \sum_{i=1}^{n} |x_i| \|Te_i\| \leq \sqrt{\sum_{i=1}^{n} x_i^2} \sqrt{\sum_{i=1}^{n} \|Te_i\|^2}$$

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$$= c\|x\|,$$

where $c = \sqrt{\sum_{i=1}^{n} \|Te_i\|^2}$. Thus $T$ is bounded.

Still assuming $T : \mathbb{R}^n \to Y$ is linear, but now further assuming also that $T$ is an isomorphism (i.e., is 1-1 onto), put

$$A = \{x \in \mathbb{R}^n : \|x\| = 1\}.$$

By the Heine-Borel Theorem, $A$ is compact in $\mathbb{R}^n$. Since $T$ is 1-1 we have $\|Tx\| > 0$ for all $x \in A$. By what we have already proved above, $T$ is bounded, hence continuous. Also, the norm function $\|\cdot\|$ on $Y$ is continuous. Thus, the continuous function $x \mapsto \|Tx\|$ has a positive minimum on $A$, so $T^{-1}$ is bounded by Lemma 11 on Page 2.

Now for the general case $T : X \to Y$, it suffices to prove (1), because then (2) will follow by applying (1) to $T^{-1}$. Let $n = \dim X$, and choose an isomorphism $U : \mathbb{R}^n \to X$. Define $S : \mathbb{R}^n \to Y$ by $S = TU$. Then $S$ is bounded by the case $X = \mathbb{R}^n$ of (1). We have $T = SU^{-1}$, and $U^{-1}$ is bounded by the case $X = \mathbb{R}^n$ of (2). Hence $T$ is bounded. \qed

Page 3, line 1 of Observation 15: replace $\cdot\|\|$ by $\|\cdot\|$ in “ and $\cdot\|\$ ”
Page 4, Corollary 18 (2): replace “converges” by “has a convergent subsequence” in “every bounded sequence in $X$ converges” (It must have been very late at night when I wrote that one.)

Page 5, item (3): replace $i$ by $j$ in “For each $i = 1, \ldots, n$”

Page 7, line 5 of proof of Proposition 27: replace $I - I$ by $I - T$ in $(I - I) \sum_{n=0}^{\infty} T^n$