Integrable functions

Notation and Terminology. Throughout this section $X$ will denote a Euclidean space.

Definition 1. Let $f$ be a measurable function.

1. The Lebesgue integral of $f$ is
   \[ \int f = \int f(x) \, dx := \int f^+ - \int f^-, \]
   provided this makes sense (that is, if the right hand side is not $\infty - \infty$).

2. $\int_A f := \int f \chi_A$ if $A \in \mathcal{M}$.

3. $f$ is integrable if $\int f \in \mathbb{R}$, and $L^1 = L^1(X)$ denotes the set of integrable functions.

Observation 2. If $\int f$ makes sense then at least one of $f^+$ and $f^-$ is integrable, hence $\int_A f$ makes sense also, for every measurable set $A$.

Observation 3. $f$ is integrable if and only if both $f^+$ and $f^-$ are.

Here’s a slightly less obvious statement:

Lemma 4. Let $f$ be measurable. Then $f$ is integrable if and only if $|f|$ is, in which case $\int |f| \leq \int f$.

Proof. Exercise. □

Although we allow integrable functions to take infinite values, we’ll soon see that for many purposes we can assume our functions are real-valued. Here’s the first step:

Observation 5. If $f \in L^1$, then $|f| < \infty$ a.e., so $f(x) \in \mathbb{R}$ a.e. $x$.

The next step will be to notice that we can ignore what happens on null sets, but first we need to know that we can chop up integrable functions:

Lemma 6. If $f$ is measurable and $A \in \mathcal{M}$ then
   \[ \int_A f = \int_A f^+ - \int_A f^-. \]

Proof. $(f \chi_A)^+ = f^+ \chi_A$ and $(f \chi_A)^- = f^- \chi_A$, so the result follows from the definitions. □

Corollary 7. If $f$ is measurable and $m(A) = 0$ then $\int_A f = 0$ and $\int_{A^c} f = \int f$.

The above corollary is an instance of “null sets don’t matter”. We’ll make bigger use of this than you might initially imagine, namely to modify the official definition of $L^1$. This will require a perhaps surprisingly fussy discussion. First of all, just to ensure that there is no confusion:
**Definition 8.** A function is *almost-everywhere-defined*, or *a.e.-defined*, if its domain has null complement.

Thus, to say \( f : A \to \mathbb{R} \) is a.e.-defined just means \( m(A^c) = 0 \). If \( f \) is a.e.-defined and measurable, then every extension of \( f \) to all of \( X \) is measurable. Moreover, if any of these extensions is integrable, then so is every other one; this motivates:

**Definition 9.** An a.e.-defined function is *integrable* if some, hence every, extension to \( X \) is integrable.

Next, we have to allow unfettered modification of functions on null sets:

**Definition 10.** Two a.e.-defined measurable functions \( f \) and \( g \) are *equivalent*, written \( f \sim g \), if \( f = g \) a.e.

It’s not hard to verify that this gives an equivalence relation\(^1\) on the set of a.e.-defined measurable functions.

**Definition 11.** We redefine \( L^1 \) to be the set of *equivalence classes\(^2\) of a.e.-defined integrable functions under the equivalence relation \( \sim \).

Thus integrable functions which agree a.e. determine the same element of \( L^1 \). However, for convenience we abuse this by continuing to speak of integrable functions as elements of \( L^1 \) — this causes no confusion in practice. Also, from now on we’ll usually drop\(^3\) the cumbersome phrase “a.e.-defined”, so that when we refer to an integrable function on \( X \) we tacitly allow that it might not be defined everywhere, just a.e. In fact, we’ll do the same for measurable functions in general. We need this flexibility because, for example, a limit of a sequence of measurable functions might not be everywhere-defined. Also, we want to add measurable functions, and there is an obstruction: if \( f \) and \( g \) are measurable, then for some \( x \) the values \( f(x) \) and \( g(x) \) could both be infinite with opposite signs, so that \( f(x) + g(x) \) would be undefined in \( \mathbb{R} \). The solution is now clear: assuming \( f(x) + g(x) \) makes sense a.e. \( x \) — for example, if \( f, g \in L^1 \) — the we get an a.e.-defined measurable function \( h \) such that \( h(x) = f(x) + g(x) \) a.e. \( x \). This is how we will add measurable functions. Suppose \( f \) and \( g \) are integrable, and choose \( A \in \mathcal{M} \) such that \( A^c \) is null and \( h(x) = f(x) + g(x) \) makes sense for all \( x \in A \). Then

\[
\int |h| = \int_A |f + g| \leq \int_A |f| + \int_A |g| = \int |f| + \int |g|,
\]

so \( h \) is integrable.

The above discussion justifies:

**Definition 12.** The sum of \( f, g \in L^1 \), denoted \( f + g \), is (the equivalence class of) any measurable function \( h \) such that \( h(x) = f(x) + g(x) \) a.e. \( x \).

With the above definition of addition in \( L^1 \), we have:

**Proposition 13.** \( L^1 \) is a vector space, and \( \int : L^1 \to \mathbb{R} \) is linear.

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\(^1\)That is, (1) \( f \sim f \), (2) if \( f \sim g \) then \( g \sim f \), and (3) if \( f \sim g \) and \( g \sim h \) then \( f \sim h \)

\(^2\)The equivalence class of \( f \) is \( \{ g : g \sim f \} \).

\(^3\)to your great relief
Proof. We saw above that $L^1$ is closed under addition. If $f \in L^1$ and $c > 0$ then $cf$ is measurable and

$$\int cf = \int (cf)^+ - \int (cf)^- = \int cf^+ - \int cf^- = c \int f^+ - c \int f^- = c \int f,$$

and similarly if $c < 0$. In particular, $cf \in L^1$. It is routine, albeit a little tedious, to verify that $L^1$ is a vector space with these operations (remember that elements of $L^1$ are equivalence classes).

It remains to check additivity of integration. Let $f, g \in L^1$ and $c \in \mathbb{R}$. We use a trick: putting $h = f + g$, we have

$$h^+ - h^- = f^+ - f^- + g^+ - g^-,$$

so

$$h^+ + f^- + g^- = h^- + f^+ + g^+,$$

hence

$$\int h^+ + \int f^- + \int g^- = \int h^- + \int f^+ + \int g^+,$$

thus

$$\int (f + g) = \int (h^+ - h^-) = \int h^+ - \int h^- = \int f^+ - \int f^- + \int g^+ - \int g^- = \int f + \int g.$$

□

The above result is only half the story; it’ll be important for us to know that in fact

**Proposition 14.** $L^1$ is a normed space with norm

$$\|f\|_1 := \int |f|,$$

and the linear functional $\int : L^1 \to \mathbb{R}$ is continuous.

Proof. First of all, if $f \in L^1$ then $\|f\|_1 = \int |f| \geq 0$, and

$$\|f\|_1 = 0 \iff \int |f| = 0 \iff |f| = 0 \text{ a.e.}$$

$$\iff f = 0 \text{ a.e.}$$

$$\iff f \text{ is the 0 element of } L^1.$$

Next, in previous computations we’ve verified $\int |cf| = |c| \int |f|$ and $\int |f + g| \leq \int |f| + \int |g|$, so $\| \cdot \|_1$ is a norm on $L^1$. Since the integral is linear on $L^1$, continuity follows from

$$\left| \int f \right| \leq \int |f| = \|f\|_1.$$  

□
Thus, another reason we need to be using equivalence classes is so that only the 0 element of \( L^1 \) has norm 0.

We’ll have to refer to convergence in the metric space \( L^1 \) a lot, and it’s customary to abbreviate this to “convergence in \( L^1 \)”; in particular, we’ll say “\( f_n \to f \) in \( L^1 \).

**Observation 15.** In the Dominated Convergence Theorem, we actually have \( f_n \to f \) in \( L^1 \).

Here is one sense in which equivalent functions “behave the same” as far as integration is concerned:

**Corollary 16.** If \( f, g \in L^1 \), then \( \int_A f = \int_A g \) for all \( A \in \mathcal{M} \) if and only if \( f = g \) a.e.

**Proof.** Replacing \( f \) by \( f - g \), without loss of generality \( g = 0 \). Put

\[
P = \{ x : f^+(x) \neq 0 \} = \{ x : f(x) > 0 \} \quad \text{and} \quad N = \{ x : f^-(x) \neq 0 \} = \{ x : f(x) < 0 \},
\]

so that \( f^+ = f\chi_P \) and \( f^- = -f\chi_N \).

Assume \( \int_A f = 0 \) for all \( A \in \mathcal{M} \). Then in particular \( \int f^+ = \int_P f = 0 \), so \( f^+ = 0 \) a.e. Similarly, \( f^- = 0 \) a.e. Thus \( f = 0 \) a.e.

Conversely, assume \( f = 0 \) a.e. Then \( m(P \cup N) = 0 \), so \( m(P) = m(N) = 0 \), hence \( f^+ \) and \( f^- \) are 0 a.e. If \( A \in \mathcal{M} \) then

\[
\int_A f = \int_A f^+ - \int_A f^- = 0 - 0 = 0.
\]

We’re now ready to complete the “triumvirate of integration theory” (the other two being Fatou’s Lemma and the Monotone Convergence Theorem), and the following result is arguably the most important of the three:

**Theorem 17** (Dominated Convergence Theorem). Let \( (f_n) \) be a sequence in \( L^1 \) converging to \( f \) a.e. Suppose there exists \( g \in L^1 \) such that for every \( n \in \mathbb{N} \) we have \( |f_n| \leq g \) a.e. Then \( f \in L^1 \) and

\[
\int f_n \to \int f.
\]

Moreover, \( f_n \to f \) in \( L^1 \).

**Proof.** Since \( f_n \to f \) a.e., \( f \) is measurable. Since \( |f_n| \leq g \) a.e. for all \( n \), for a.e. \( x \) we have \( |f_n(x)| \leq g(x) \) for all \( n \), hence \( |f(x)| \leq g(x) \). Thus \( |f| \leq g \) a.e., so \( f \in L^1 \).

For the other part, we need a trick: \( g + f_n \geq 0 \) for all \( n \), so by Fatou’s Lemma

\[
\int g + \int f = \int (g + f) = \int \lim (g + f_n)
\]

\[
\leq \lim \inf \int (g + f_n) = \lim \inf \left( \int g + \int f_n \right) = \int g + \lim \inf \int f_n,
\]

hence

\[
\int f \leq \lim \inf \int f_n.
\]
Applying this to $-f$, we get

$$-\int f = \int (-f) \leq \liminf \int (-f_n) = \liminf \int f_n = - \limsup \int f_n,$$

so

$$\int f \geq \limsup \int f_n.$$

Therefore we must have

$$\limsup \int f_n = \liminf \int f_n = \int f.$$

Finally, we have $|f_n - f| \to 0$ a.e., and $|f_n - f| \leq 2g$ a.e., so by the above we have

$$\|f_n - f\|_1 = \int |f_n - f| \to 0.$$

Just as the Monotone Convergence Theorem has a series version, so does the Dominated Convergence Theorem:

**Corollary 18 (Series Version of DCT).** If $\sum f_n$ is a series in $L^1$ such that $\sum \int |f_n| < \infty$, then the series $\sum f_n$ converges a.e. and in $L^1$, and

$$\int \sum f_n = \sum \int f_n.$$

**Proof.** By the Monotone Convergence Theorem

$$\int \sum |f_n| = \sum \int |f_n| < \infty,$$

so $\sum |f_n| < \infty$ a.e., hence $\sum f_n$ converges a.e. For each $k \in \mathbb{N},$

$$\sum_{1}^{k} f_n \leq \sum_{1}^{k} |f_n| \leq \sum_{1}^{\infty} |f_n|,$$

and the latter function is integrable, so by the Dominated Convergence Theorem $\sum_{1}^{\infty} f_n \in L^1$ and

$$\int \sum_{1}^{\infty} f_n = \lim_{k} \int \sum_{1}^{k} f_n = \lim_{k} \sum_{1}^{k} \int f_n = \sum_{1}^{\infty} \int f_n.$$

Finally, the series $\sum f_n$ actually converges in $L^1$ by the same reasoning as in the Dominated Convergence Theorem. \qed

When we apply the above result, we’ll sometimes just say “by the Dominated Convergence Theorem”.

**Corollary 19.** $L^1$ is a Banach space.

**Proof.** It suffices to observe that, by Corollary 18, every absolutely convergent series $\sum f_n$ in $L^1$ converges. \qed

Here’s an unexpected bonus:

**Corollary 20.** If $f_n \to f$ in $L^1$ then some subsequence of $(f_n)$ converges to $f$ a.e.
Proof. The proof of the result that a normed space is complete if every absolutely convergent series converges proceeded by replacing a given Cauchy sequence, such as \((f_n)\), by a subsequence \((f_{n_k})\) comprising the partial sums of an absolutely convergent series. By the Dominated Convergence Theorem, every absolutely convergent series in \(L^1\) also converges a.e. Thus \(f_{n_k} \to f\) a.e. □

Definition 21.

(1) The support of a continuous function \(f : X \to \mathbb{R}\) is
\[
\text{supp } f = \{ x \in X : f(x) \neq 0 \}.
\]

(2) \(C_c(X)\) denotes the set of continuous real-valued functions on \(X\) with compact support.

Proposition 22. \(C_c(X)\) is dense in \(L^1\).

Proof. Since:

- the integrable simple functions are dense in \(L^1\),
- the characteristic functions of sets in \(\mathcal{E}\) are dense in the integrable characteristic functions, and
- every set in \(\mathcal{E}\) is a finite disjoint union of boxes,

it suffices to show that if \(B\) is a box then \(\chi_B\) can be approximated in the \(L^1\) norm by an element of \(C_c(X)\). Without loss of generality \(B\) has positive measure, say \(B = \prod_{j=1}^n I_j\) for nondegenerate intervals \(I_1, \ldots, I_n\). Temporarily fix \(c \in (0, 1)\), and for each \(j\) let \(I'_j\) be the an interval concentric with \(I_j\) such that
\[
|I'_j| = c|I_j|,
\]
and then put \(B' = \prod_{j=1}^n I'_j\). Then
\[
m(B') = c^n m(B).
\]

Now for each \(j\) let \(f_j\) be the continuous piecewise-linear\(^4\) function on \(\mathbb{R}\) which is 0 outside of \(I_j\), 1 on \(I'_j\), and linear on each of the two gaps between \(I_j\) and \(I'_j\). Then define \(f : X \to \mathbb{R}\) by
\[
f(x_1, \ldots, x_n) = \prod_{j=1}^n f_j(x_j).
\]

Then \(f\) is continuous, \(0 \leq f \leq 1\), \(f\) is 0 outside \(B\), and \(f\) is 1 on \(B'\). In particular, \(f\) has compact support. Since \(\chi_{B'} \leq f \leq \chi_B\), we have
\[
\|\chi_B - f\|_1 = \int (\chi_B - f) \leq \int (\chi_B - \chi_B') = m(B) - m(B')
= (1 - c^n)m(B) \xrightarrow{c 
\to 0} 0.
\]

Corollary 23 (Lebesgue’s Characterization of Riemann Integrability). A bounded real-valued function \(f\) on \([a, b]\) is Riemann integrable if and only if it is continuous a.e. Moreover, in this case \(f\) is also Lebesgue integrable, and the Riemann and Lebesgue integrals of \(f\) coincide.

\(^4\)i.e., it’s graph is a finite union of line segments
Proof. For each \( n \in \mathbb{N} \) let \( \mathcal{P}_n \) be the family of \( 2^n \) nonoverlapping\(^5\) closed intervals of length \( 2^{-n}(b - a) \) with union \([a, b]\). Define \( g_n, h_n : [a, b] \to \mathbb{R} \) as follows: for \( x \in [a, b] \), first find \( I \in \mathcal{P}_n \) such that \( x \in I \). Then define:

- \( g_n(x) = h_n(x) = f(x) \) if \( x \in \partial I \);
- \( g_n(x) = \inf_{x \in I} f(x) \) and \( h_n(x) = \sup_{x \in I} f(x) \) if \( x \in I^o \).

Then \( g_1 \leq g_2 \leq \cdots \leq f \leq \cdots \leq h_2 \leq h_1 \), and by Darboux’s Theorem \( f \) is Riemann integrable if and only if \( \lim_n \int g_n = \lim_n \int h_n \), in which case the Riemann integral of \( f \) is this common limit.

Put \( g = \lim g_n \) and \( h = \lim h_n \). By the Dominated Convergence Theorem, \( g \) and \( h \) are integrable and

\[
\int (h - g) = \lim \int (h_n - g_n) = \lim \int h_n - \lim \int g_n.
\]

Since \( h - g \geq 0 \), we have

\[
\int (h - g) = 0 \iff h - g = 0 \text{ a.e.} \iff g = h \text{ a.e.}
\]

Put \( D = \bigcup_{n=1}^{\infty} \bigcup_{I \in \mathcal{P}_n} \partial I \), a countable subset of \([a, b]\). Claim: for \( x \in [a, b] \setminus D \), \( f \) is continuous at \( x \) if and only if \( g(x) = h(x) \). To see this, first assume continuity, and, given \( \varepsilon > 0 \), choose \( \delta > 0 \) such that for all \( y \in [a, b] \), if \( |y - x| < \delta \) then \( |f(y) - f(x)| < \varepsilon \). Choose \( n \) such that \( 2^{-n}(b - a) < \delta \), and a find \( I \in \mathcal{P}_n \) such that \( x \in I^o \). Then

\[
|f(y) - f(z)| < 2\varepsilon \quad \text{for all } y, z \in I,
\]

so

\[
0 \leq h_n(x) - g(x) \leq h_n(x) - g_n(x) \leq 2\varepsilon.
\]

Letting \( \varepsilon \to 0 \), we get \( g(x) = h(x) \). For the converse direction, we essentially reverse the steps: assuming \( g(x) = h(x) \), and given \( \varepsilon > 0 \), there exists \( n \) such that \( h_n(x) - g_n(x) < \varepsilon \), and a subinterval \( I \) associated to \( P_n \) such that \( x \in I^o \). Let \( \delta \) be the minimum of the distances from \( x \) to the two endpoints of \( I \). Then

\[
|f(x) - f(y)| < \varepsilon \quad \text{if } |x - y| < \varepsilon,
\]

and we’ve shown \( f \) is continuous at \( x \).

Thus,

- \( f \) is Riemann integrable
- \( \iff g = h \) a.e.
- \( \iff g = h \) a.e. on \( P^c \) (since \( m(P) = 0 \))
- \( \iff f \) is continuous a.e. on \( P^c \)
- \( \iff f \) is continuous a.e.

\(^5\)i.e., intersecting at most at endpoints
For the other part, if these equivalent conditions are satisfied, then \( f = g \) a.e. because \( g \leq f \leq h \). Thus \( f \in L^1 \) and

\[
\int f = \int g = \lim \int g_n,
\]

which coincides with the Riemann integral of \( f \). \qed