Measures

We will develop the theory of Lebesgue integration in $\mathbb{R}^n$, not just $\mathbb{R}$. One feature of this theory is the extensive use of sequences. To avoid the crushing loss of the letter “$n$” in sequences, we will frequently use:

**Definition 1.** A **Euclidean space** is any space of the form $\mathbb{R}^n$ for some $n \in \mathbb{N}$. We allow the norm to be arbitrary, although frequently, and by default unless otherwise specified, we will use the Euclidean norm $\|x\| = (\sum_{i=1}^{n} x_i^2)$.

The development of the theory will require us to “measure” subsets of Euclidean spaces. The crucial first step is to realize that we cannot hope to measure all subsets in a reasonable way (and this will become clearer as we develop the theory of Lebesgue measure). Consequently, we have to suitably limit the domains of our measures:

**Definition 2.** A **$\sigma$-algebra** on a set $X$ is a nonempty family of subsets of $X$ which is closed under complements and countable unions.

Thus, if $A$ is a $\sigma$-algebra on $X$, then $A^c \in A$ for all $A \in A$, and $\bigcup_{n=1}^{\infty} A_n \in A$ if $\{A_n\} \subset A$.

**Observation 3.** If $A$ is a $\sigma$-algebra on $X$, then:

1. $\emptyset, X \in A$.
2. $A$ is closed under differences and countable intersections.
3. Every countable union in $A$ (that is, of elements of $A$) can be expressed as a countable disjoint union in $A$. (Just replace $A_n$ by $A_n \setminus \bigcup_{i<n} A_i$.)

When showing that a family of sets is a $\sigma$-algebra, the most common way to verify that the family is nonempty is to show it contains the empty set.

**Example 4.** The biggest $\sigma$-algebra on $X$ is the family of all subsets of $X$, and the smallest is $\{X, \emptyset\}$.

**Observation 5.** The intersection of any family of $\sigma$-algebras is a $\sigma$-algebra.

**Definition 6.** Let $S$ be a family of subsets of $X$. The **$\sigma$-algebra generated by $S$** is the intersection of all $\sigma$-algebras on $X$ containing $S$.

**Example 7.** The countable-cocountable $\sigma$-algebra on $X$ is the family of all sets which are either countable or cocountable (i.e., have countable complement), and is generated by the family of singletons.

**Definition 8.** A **measure** on a $\sigma$-algebra $A$ is a function $\mu : A \to [0, \infty]$ such that:

1. $\mu(\emptyset) = 0$, and
2. $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ if $\{A_n\} \subset A$ is disjoint.
Definition 9. Property (2) in the above definition is countable additivity of \( \mu \). Occasionally it is convenient to refer to a weaker property: additivity, which means \( \mu(\bigcup_{i=1}^{k} A_i) = \sum_{i=1}^{k} \mu(A_i) \) whenever \( A_1, \ldots, A_k \) are disjoint in \( \mathcal{A} \). Sometimes additivity is called finite additivity to emphasize that \( \mu \) is only additive on finite families of sets.

To motivate the use of extended real numbers, remember that our main objective is to measure sets in \( \mathbb{R}^n \); in the case of \( \mathbb{R} \), the measure of an interval should be its length. For convenience, we want to be able to measure the whole space — so we have to allow the measure to take the value \( \infty \).

Lemma 10. If \( \mu \) is a measure on \( \mathcal{A} \), then:

1. \( \mu(A) \leq \mu(B) \) if \( A \subset B \).
2. \( \mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i) \) if \( A_1, A_2, \ldots \in \mathcal{A} \).

Proof. Exercise. \( \square \)

Definition 11. (1) above is monotonicity, and (2) is countable subadditivity. Occasionally it is convenient to refer to a weakening of (2): subadditivity, which means \( \mu(\bigcup_{i=1}^{k} A_i) \leq \sum_{i=1}^{k} \mu(A_i) \) whenever \( A_1, \ldots, A_k \in \mathcal{A} \).

A central theme of measure theory is that sets of measure zero can be ignored; it will be convenient to have a name for these sets:

Definition 12. A null set is a set of measure zero.

Corollary 13. If \( \mu \) is a measure on \( \mathcal{A} \), then the family of null sets in \( \mathcal{A} \) is closed under countable unions.

Proposition 14. Let \( \mu \) be a measure on \( \mathcal{A} \), and let \( A_1, A_2, \ldots \in \mathcal{A} \).

1. (Continuity from Below) If \( A_1 \subset A_2 \subset \cdots \), then \( \mu(A_i) \to \mu(\bigcup_{i} A_i) \).
2. (Continuity from Above) If \( A_1 \supset A_2 \supset \cdots \) and \( \mu(A_1) < \infty \), then \( \mu(A_i) \to \mu(\bigcap_{i} A_i) \).

Proof. (1) Put

\[
B_i = \begin{cases} 
A_1 & \text{if } i = 1 \\
A_i \setminus A_{i-1} & \text{if } i = 2, 3, \ldots
\end{cases}
\]

Then \( B_1, B_2, \ldots \in \mathcal{A} \) are disjoint, \( A_i = \bigcup_{1}^{i} B_k \), and \( \bigcup_{1}^{\infty} A_i = \bigcup_{1}^{\infty} B_i \). Thus

\[
\mu(A_i) = \mu(\bigcup_{1}^{i} B_k) = \sum_{1}^{i} \mu(B_k) \\
\to \sum_{1}^{\infty} \mu(B_i) = \mu(\bigcup_{1}^{\infty} B_i) = \mu(\bigcup_{1}^{\infty} A_i).
\]
(2) Put $B_i = A_1 \setminus A_i$. Then $B_1 \subset B_2 \subset \cdots \subset A_1$ and $\bigcup_i B_i = A_1 \setminus \bigcap_i A_i$, so

$$\mu(A_i) = \mu(A_1 \setminus B_i) = \mu(A_1) - \mu(B_i) \quad \text{since } \mu(A_1) < \infty$$

$$\rightarrow \mu(A_1) - \mu\left(\bigcup_i B_i\right) = \mu\left(A_1 \setminus \bigcup_i B_i\right) = \mu\left(\bigcap_i A_i\right). \quad \square$$

**Observation 15.** *In Continuity from Above, it is enough for some $A_i$ to have finite measure.*