Inverse functions

**Notation and Terminology.** Thoughout this section \( X \) will denote a finite-dimensional normed space.

The Inverse Function Theorem from 1-variable calculus says that if \( U \) is an open interval in \( \mathbb{R} \), \( f : U \to \mathbb{R} \) is 1-1 and continuous, \( a \in U \), and \( f \) is differentiable at \( a \) with \( f'(a) \neq 0 \), then the inverse \( f^{-1} \) is differentiable at \( a \) and

\[
(f^{-1})'(f(a)) = \frac{1}{f'(a)}.
\]

The higher-dimensional version (Theorem 2 below) requires a much stronger assumption on the derivative of \( f \): it must exist and in fact be continuous on some neighborhood of \( a \). As a consolation, we can deduce that \( f \) is 1-1 near \( a \). But before we get into the theorem itself, let’s observe that the (appropriate higher-dimensional version of the) formula for the derivative of the inverse follows immediately from the Chain Rule:

**Proposition 1.** Let \( E \subset X \), \( f : E \to X \), and \( a \in E \). Assume that \( f \) is 1-1, so that we have an inverse function \( f^{-1} : f(E) \to E \). If \( f \) is differentiable at \( a \) and \( f^{-1} \) is differentiable at \( f(a) \), then:

1. \( f'(a) \) is invertible, and
   \[
   f'(a)^{-1} = (f^{-1})'(f(a));
   \]
2. If \( f' \) is continuous at \( a \) then \((f^{-1})'\) is continuous at \( f(a) \).

**Proof.** Exercise. \( \square \)

**Theorem 2** (Inverse Function Theorem). Let \( E \subset X \), \( f : E \to X \), and \( a \in E \). If \( f \) is \( C^1 \) and \( f'(a) \) is invertible, then there exist open sets \( U, V \subset X \) such that \( a \in U \subset E \) and \( f : U \to V \) is 1-1 onto with \( C^1 \) inverse.

**Proof.** We can simplify things considerably with some preliminaries. We will choose \( U \) in such a way that \( f'(x) \) is invertible for all \( x \in U \). Consequently, to conclude that \( f(U) \) is open, it will suffice to show that \( f(a) \in f(E) \), and for this it is enough to have \( f(a) \in f(U) \) for some \( U \subset E \). For the same reason, once we have such a set \( U \), to see that \( f^{-1} \) is \( C^1 \) on \( f(U) \) it will suffice to show that \( f^{-1} \) is differentiable at \( f(a) \), because Proposition 1 will then imply that \((f^{-1})'\) is continuous at \( f(a) \), and this holds for every \( a \) at which the derivative of \( f \) is invertible, hence at \( f(x) \) for every element \( x \in U \).

Replacing \( f \) by \( f'(a)^{-1} \circ f \), without loss of generality \( f'(a) = I \). Then replacing \( f \) by \( x \mapsto f(x + a) - f(a) \), without loss of generality \( a = 0 \) and \( f(0) = 0 \).
Since $f$ is $C^1$, there exists $\varepsilon > 0$ such that $U := B_\varepsilon(0) \subset E$ and
\[ \|f'(x) - I\| < \frac{1}{2} \quad \text{for all } x \in U. \]
In particular, $f'(x)$ is invertible for all $x \in U$.

Since $U$ is convex and open, the Mean Value Inequality tells us that for all $x, z \in U$ we have
\[ \|(f - I)(x) - (f - I)(z)\| \leq \frac{\|x - z\|}{2}, \]

hence
\[ (1) \quad \|f(x) - f(z)\| \geq \frac{\|x - z\|}{2}. \]

Thus $f$ is 1-1 on $U$, and the inequality (1) also shows that $f^{-1}$ is continuous on $V := f(U)$.

For the rest of the proof we ignore the original $E$ and regard $f$ as a 1-1 function from $U$ onto $V$.

As we discussed at the start of the proof, it remains to show that $0 \in V^\circ$ and $f^{-1}$ is differentiable at 0.

For the first, put $B = B_{\varepsilon/2}(0)$, which is an open ball containing 0 such that $\overline{B} \subset U$. We have $0 \notin f(\partial B)$ because $0 \notin \partial B$ and $f$ is 1-1 on $U$. Since $\partial B$ is compact and $z \mapsto \|f(z)\|$ is continuous, there exists $\delta > 0$ such that
\[ \|f(z)\| \geq 2\delta \quad \text{for all } z \in \partial B. \]

Claim: $B_\delta(0) \subset V$. Let $w \in B_\delta(0)$, and define $g : \overline{B} \to \mathbb{R}$ by $g(z) = \|f(z) - w\|^2$. Since $\overline{B}$ is compact and $g$ is continuous, $g$ has a minimum at some $t \in \overline{B}$. Since $g(0) = \|w\|^2 < \delta^2$ and $g(z) \geq \delta^2$ for all $z \in \partial B$, we have $t \in B$. Thus for all $y \in X$ we have
\[ 0 = g'(t)y = \langle 2(f(t) - w), f'(t)y \rangle. \]

Since $f'(t)$ is invertible, there exists $y \in X$ such that $f'(t)y = f(t) - w$, so we must have $f(t) - w = 0$, proving the claim.

It remains to show that $f^{-1}$ is differentiable at 0, and in fact we will show that $(f^{-1})'(0) = I$.

Let $y \in V \setminus \{0\}$ and $x = f^{-1}(y)$. Then $x \in U \setminus \{0\}$, so
\[
\frac{\|f^{-1}(y) - f^{-1}(0) - Iy\|}{\|y\|} \leq \frac{\|x\|}{\|y\|} \frac{\|x - f(x)\|}{\|x\|} \leq 2 \frac{\|f(x) - x\|}{\|x\|},
\]

since $\|y\| \geq \|x\|/2$ by (1). Since $f^{-1}$ is continuous, as $y \to 0$ we have $x \to 0$, hence $\|f(x) - x\|/\|x\| \to 0$ because $f'(0) = I$. \qed

Continuity of the derivative was crucial in the Inverse Function Theorem. We only assumed $f'$ was invertible at one point, but then by continuity we knew $f'$ must be invertible nearby. The theorem would definitely become false if we don’t assume $f'$ is continuous, even in the 1-variable case:
Example 3. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x + 2x^2 \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then $f$ is differentiable on $\mathbb{R}$ and $f'(0) \neq 0$, but $f$ is not 1-1 on any open interval containing 0.

Example 4. Let $U = \{(x, y) \in \mathbb{R}^2 : y \neq 0\}$, and define $f : U \to \mathbb{R}^2$ by

$$f(x, y) = \left( e^x + xy^2, \frac{2 \sin \pi x}{y} \right).$$
The derivative $f'(x, y)$ is represented by the matrix

$$\begin{pmatrix} e^x + y^2 & 2xy \\ 2\pi \cos \pi x & -2\sin \pi x \\ y & y^2 \end{pmatrix},$$
which is continuous on $U$, so $f$ is $C^1$. At $(x, y) = (1, 1)$, the matrix is

$$\begin{pmatrix} e + 1 & 2 \\ -2\pi & 0 \end{pmatrix},$$
which is invertible. Thus the Inverse Function Theorem tells us $f$ is invertible near $(1, 1)$, and moreover the derivative of the inverse function at $f(1, 1) = (e + 1, 0)$ is represented by

$$\begin{pmatrix} e + 1 & 2 \\ -2\pi & 0 \end{pmatrix}^{-1} = \frac{1}{4\pi} \begin{pmatrix} 0 & -2 \\ 2\pi & e + 1 \end{pmatrix}.$$

On the other hand, $f'(0, 1)$ is represented by

$$\begin{pmatrix} 2 & 0 \\ 2\pi & 0 \end{pmatrix},$$
which is not invertible. Thus the Inverse Function Theorem does not apply at this point, and so in particular tells us nothing about whether $f$ is invertible near $(0, 1)$.

It follows from the Inverse Function Theorem that if $f$ is $C^1$ and $f'(x)$ is invertible for all $x$ in the domain of $f$, then the range of $f$ is open. The Inverse Function Theorem only tells us $f$ is locally 1-1; it need not be 1-1:

Example 5. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x, y) = \left( e^x \cos y, e^x \sin y \right).$$

Then $f$ is $C^1$ and $f'(x, y)$ is invertible for all $(x, y) \in \mathbb{R}^2$, but $f$ is not 1-1. The Inverse Function Theorem does guarantee that the range of $f$ is open; in fact it is the set $\mathbb{R}^2 \setminus \{0\}$. 