Integrable functions — Exercises

1. Let $f$ be measurable. Prove that $f$ is integrable if and only if $|f|$ is, in which case $\int |f| \leq \int |f|$.

2. Translation-Invariance of Lebesgue Integration. Let $c \in X$. Prove that if $f \in L^1$, then so is $x \mapsto f(x + c)$, and

$$\int f(x + c) \, dx = \int f.$$

3. Prove that there is no continuous function equal to $\chi_{[0,1]}$ a.e. on the interval $[0,2]$.

4. Let $T$ be a metric space and $A \subset T$, and $f : X \times A \to \mathbb{R}$. Suppose $f(\cdot, t) \in L^1$ for each $t \in A$, and define $h : A \to \mathbb{R}$ by

$$h(t) = \int f(x, t) \, dx.$$

Let $s$ be a cluster point of $A$, and assume that

$$l(x) := \lim_{t \to s} f(x, t)$$

exists for every $x \in X$. Further assume that there is an integrable function $g$ on $X$ such that

$$|f(x, t)| \leq g(x) \quad \text{for all } (x, t) \in X \times A.$$

Prove that $l \in L^1$ and

$$\int l = \lim_{t \to s} \int h(t).$$

5. Let $f : \mathbb{R} \times (a, b) \to \mathbb{R}$. Assume that $f(\cdot, y)$ is integrable for all $y \in (a, b)$, and define $F : (a, b) \to \mathbb{R}$ by

$$F(y) = \int f(x, y) \, dx.$$

Assume also that $D_2f$ exists on $\mathbb{R} \times (a, b)$ and there exists $g \in L^1(\mathbb{R})$ such that

$$|D_2f(x, y)| \leq g(x) \quad \text{for all } (x, y) \in \mathbb{R} \times (a, b).$$

Prove that $F$ is differentiable and

$$F'(y) = \int D_2f(x, y) \, dx \quad \text{for all } y \in (a, b).$$

6. Find and justify:

$$\lim_{k \to \infty} \int_0^\infty \frac{k \sin(x/k)}{x(1 + x^2)} \, dx.$$
7. Let $f \in L^1$, and let $A_1, A_2, \ldots \in \mathcal{M}$ be disjoint. Prove that

$$\int_{\bigcup_{i=1}^{\infty} A_n} f = \sum_{i=1}^{\infty} \int_{A_n} f.$$ 

8. Prove that if $m(A_n) < \infty$ for all $n \in \mathbb{N}$ and $\chi_{A_n} \to f$ in $L^1$, then $f$ is a.e. equal to a characteristic function.

9. **Absolute Continuity.** Let $f \in L^1$. Prove that for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left|\int_{A} f\right| < \varepsilon \quad \text{if} \ m(A) < \delta.$$ 

Hint: consider a sequence $(A_n)$ in $\mathcal{M}$ with $m(A_n) \to 0$.

10. Prove that the following sets are dense in $L^1$:

(a) the set of integrable simple functions;

(b) the set of integrable bounded functions.

11. Prove that the set of integrable functions which vanish outside a bounded set\(^2\) is dense in $L^1$.

12. Let $A$ have finite measure, and let $\varepsilon > 0$.

(a) Prove that there exists $B \in \mathcal{E}$ such that $\|\chi_A - \chi_B\|_1 < \varepsilon$.

(b) Prove that if $X = \mathbb{R}$ then the $B$ of part (a) can be taken as a finite disjoint union of open intervals.

13. Prove that the step functions are dense in $L^1(\mathbb{R})$.

14. Let $f \in L^1(\mathbb{R})$, and suppose $\int_0^t f = 0$ for all $t \in \mathbb{R}$. Prove that $f = 0$ almost everywhere. Hint: sets of finite measure can be approximated by finite unions of bounded intervals.

15. **Continuity of Translation.** Let $f \in L^1$, and define $g : X \to L^1$ by

$$g(t)(x) = f(x + t).$$ 

Also let $s \in X$. In this problem you’ll follow a prescribed strategy to prove that

(1) $$\lim_{t \to s} g(t) = g(s) \quad \text{in} \ L^1.$$ 

(a) Use the Monotone Convergence Theorem to prove (1) if $f \in C_c(X)$. Hint: when letting $t \to s$, it suffices to consider $t \in B = B_1(s)$,

(b) Use part (a) and density of $C_c(X)$ in $L^1$ to deduce (1) in general.

\(^1\)i.e., are zero
\(^2\)which can depend upon the function
16. **Riemann-Lebesgue Lemma.** For each \( f \in L^1(\mathbb{R}) \) define \( \Psi(f) : \mathbb{R} \to \mathbb{R} \) by
\[
\Psi(f)(t) = \int_{-\infty}^{\infty} f(x) \sin(xt) \, dx.
\]

(a) Prove that if \( f_n \to f \) in \( L^1 \) then \( \Psi(f_n) \to \Psi(f) \) uniformly on \( \mathbb{R} \).

(b) Prove that for every \( f \in L^1(\mathbb{R}) \),
\[
\lim_{t \to \infty} \Psi(f)(t) = 0.
\]

Hint: first take \( f \) to be the characteristic function of a bounded interval. You may use elementary integration formulas from calculus.

17. (a) Is the characteristic function of the Cantor set Riemann integrable?

(b) Is the characteristic function of a fat Cantor set Riemann integrable?

Give full justifications.