Integrating nonnegative functions — Exercises

1. Let $\phi \geq 0$ be simple and $c \in X$. Prove that
   \[ \int \phi(x + c) \, dx = \int \phi. \]

2. Let $\phi \geq 0$ be simple. Prove that $\int \phi = 0$ if and only if $\phi = 0$ a.e.

3. Let $f \in L^+$ and $c \in X$. Prove that
   \[ \int f(x + c) \, dx = \int f. \]

4. Borel-Cantelli Lemma. Let $A_1, A_2, \ldots \in \mathcal{M}$, with $\sum_n m(A_n) < \infty$. Prove that almost every point is in only finitely many $A_n$'s. That is, show that there exists $B \subset X$ such that $m(B) = 0$ and for all $x \notin B$, the set \( \{n \in \mathbb{N} : x \in A_n\} \) is finite.