Borel sets — Exercises

1. Let $A \in \mathcal{M}$. Prove that
   
   $$m(A) = \sup\{m(K) : K \text{ compact, } K \subset A\}.$$ 

2. Let $A \subset \mathbb{R}^n$, and suppose that for all $\epsilon > 0$ there exist a closed set $C$ such that $C \subset A$ and $m^*(A \setminus C) < \epsilon$. Prove that $A$ is measurable.

3. Define an equivalence relation $\sim$ on $\mathbb{R}$ by

   $$x \sim y \text{ if and only if } x - y \in \mathbb{Q}.$$ 

   By the Axiom of Choice, there exists a set $A$ consisting of exactly one element of each equivalence class. Then $\mathbb{R} = \bigcup_{x \in \mathbb{Q}} (A + x)$, and for distinct $x, y \in \mathbb{Q}$ we have $(A + x) \cap (A + y) = \emptyset$. You do not have to prove any of this so far.

   (a) Prove that $m^*(A) > 0$.

   (b) Prove that if $E \subset A$ and $E$ is measurable, then $m(E) = 0$. Hint: consider compact subsets of $E$.

   (c) Prove that if $B$ is a measurable subset of $\mathbb{R}$ with $m(B) > 0$, then $B$ has a nonmeasurable subset. Hint:

   $$B = \bigcup_{x \in \mathbb{Q}} (B \cap (A + x)),$$

   where $A$ is the same as above.