Boxes — Exercises

1. Prove that every open subset of $\mathbb{R}^n$ is a countable union of open boxes. Hint: consider open cubes with rational centers and sides.

2. Prove that every open subset of $\mathbb{R}^n$ is a countable union of closed cubes with disjoint interiors. Hint: consider for each $j \in \mathbb{N}$ the family $\mathcal{C}_j$ of closed cubes of side $2^{-j}$ with vertices in $(2^{-j}\mathbb{Z})^n$.

3. Prove that every open subset of $\mathbb{R}^n$ is a countable disjoint union of boxes. Hint: consider for each $j \in \mathbb{N}$ the family $\mathcal{D}_j$ of boxes of the form 
\[ \{x \in \mathbb{R}^n : a_i \leq x_i < a_i + 2^{-j}, i = 1, \ldots, n\} \] for some $a \in (2^{-j}\mathbb{Z})^n$. 