Inverse functions — Exercises

1. Let $E \subset \mathbb{R}^n$, $f : E \to \mathbb{R}^n$, and $a \in E^\circ$. Assume $f$ is differentiable at $a$. Also assume $f$ is 1-1, so that we have an inverse function $f^{-1} : f(E) \to E$. Finally, assume $f^{-1}$ is differentiable at $f(a)$.

   (1) Prove that $f'(a)$ is invertible and
   $$(f^{-1})'(f(a)) = f'(a)^{-1}.$$  

   (2) Carefully explain how part (a) implies that if $f'$ is continuous at $a$ then $(f^{-1})'$ is continuous at $f(a)$.

2. Define $f : \mathbb{R} \to \mathbb{R}$ by
   $$f(x) = \begin{cases} x + 2x^2 \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

   Show that $f$ is differentiable on $\mathbb{R}$ and $f'(0) \neq 0$, but $f$ is not 1-1 on any open interval containing 0.

3. Let $U = \{(x, y) \in \mathbb{R}^2 : y \neq 0\}$, and define $f : U \to \mathbb{R}^2$ by
   $$f(x, y) = \left( e^x + xy^2, \frac{2\sin \pi x}{y} \right).$$

   Note that $f(1, 1) = (e + 1, 0)$.

   (1) Use the Inverse Function Theorem to show that $f$ is invertible near $(1, 1)$, and find a formula for (the matrix representing) $(f^{-1})'(e + 1, 0)$.

   (2) Why does the Inverse Function Theorem not apply to the question of whether $f$ is invertible near $(0, 1)$?

4. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by
   $$f(x, y) = (e^x \cos y, e^x \sin y).$$

   Prove:
   (1) $f$ is $C^1$;
   (2) $f'(x, y)$ is invertible for all $(x, y) \in \mathbb{R}^2$;
   (3) $f$ is not 1-1;
   (4) The range of $f$ is $\mathbb{R}^2 \setminus \{0\}$. 