Mean value — Exercises

1. Verify that the function \( f : \mathbb{R}^3 \to \mathbb{R}^2 \) defined by
   \[
   f(x, y, z) = \left( \frac{2x}{\log(z^2 + 1)} e^{-x^2} \sin yz \right)
   \]
   is \( C^1 \).

2. Define \( f : \mathbb{R} \to \mathbb{R}^2 \) by \( f(x) = (\cos x, \sin x) \). Prove that there does not exist \( z \) between 0 and \( 2\pi \) such that
   \[
   f(2\pi) - f(0) = 2\pi f'(z).
   \]

3. Define \( f : \mathbb{R} \to \mathbb{R} \) by
   \[
   f(x) = \begin{cases} 
   x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\
   0 & \text{if } x = 0.
   \end{cases}
   \]
   Prove that \( f \) is differentiable on \( \mathbb{R} \), but \( f' \) is discontinuous at 0.

4. Let \( U = \{(x, y) \in \mathbb{R}^2 : y \neq 0\} \), and define \( q : U \to \mathbb{R} \) by
   \[
   q(x, y) = \frac{x}{y}.
   \]
   (a) Use the partial derivatives to help show that \( q \) is differentiable.
   (b) Find a formula for \( q'(a, b)(x, y) \) for \( (a, b) \in U, (x, y) \in \mathbb{R}^2 \).
   (c) Use the formula you found in part (b) together with the Chain Rule to derive a “quotient rule” for
   \[
   \left( \frac{f}{g} \right)'(a)x
   \]
   where \( f, g : E \to \mathbb{R}, E \subset X, a \in E^o \), both \( f \) and \( g \) are differentiable at \( a \), and \( 0 \notin \text{ran } g \).

5. Let \( U \subset \mathbb{R}^n \) be open, and let \( f : U \to \mathbb{R} \). Suppose every partial derivative \( D_i f \) is bounded on \( U \). Prove that \( f \) is continuous on \( U \). Hint: the Mean Value Theorem itself does not apply directly — instead, you should use the technique of the proof.

6. Let \( U \subset X \) be open and connected and \( f : U \to Y \) be differentiable. Prove that if \( f'(x) = 0 \) for all \( x \in U \) then \( f \) is constant.
   Hint: Fix \( a \in U \), and consider the set \( V := \{x \in U : f(x) = f(a)\} \).