Linear maps — Exercises

1. Fix \( x \in \mathbb{R}^n \), and define \( S \in B(\mathbb{R}, \mathbb{R}^n) \) and \( T \in B(\mathbb{R}^n, \mathbb{R}) \) by \( S(t) = tx \) and \( T(y) = \langle x, y \rangle \). Prove that 

\[
\|S\| = \|T\| = \|x\|.
\]

2. Let \( T \in B(X, Y) \) be onto. Prove that if \( U \subset X \) is open then so is \( T(U) \).

3. (a) Let \( Z \) be a subspace of \( X \) with the same dimension as \( Y \). Prove that the set 

\[
\{T \in B(X, Y) : T(Z) = Y\}
\]

is open in \( B(X, Y) \).

(b) Let \( F \subset X \) be linearly independent and have the same number of elements as the dimension of \( Y \). Prove that the set 

\[
\{T \in B(X, Y) : T(F) \text{ is linearly independent}\}
\]

is open in \( B(X, Y) \).

(c) Prove that the set 

\[
\{T \in B(X, Y) : T \text{ is onto}\}
\]

is open in \( B(X, Y) \).

4. Let \( m < n \), and let 

\[
1 \leq j_1 < j_2 < \cdots < j_m \leq n.
\]

Prove that 

\[
\{A \in M_{m\times n} : A_{j_1}, \ldots, A_{j_m} \text{ are linearly independent}\}
\]

is open in \( M_{m\times n} \), where \( A_j \) denotes the \( j \)th column of \( A \).

5. As we have seen, the set of bilinear maps can be identified with \( B(X, B(Y, Z)) \) via 

\[
f(x)(y) = f(x, y),
\]

hence becomes a finite-dimensional normed space, with vector space operations 

\[
(f + g)(x, y) = f(x, y) + g(x, y)
\]

\[
(cf)(x, y) = cf(x, y) \quad \text{for } c \in \mathbb{R}.
\]

In this problem you will give two characterizations of the norm on bilinear maps: if \( f : X \times Y \to Z \) is bilinear, prove:

(a) \( \|f\| = \sup \{\|f(x, y)\| : \|(x, y)\| \leq 1\} \).

(b) \( \|f\| = \min \{c \in \mathbb{R} : \|f(x, y)\| \leq c \|x\| \|y\| \text{ for all } (x, y) \in X \times Y\} \).