17. Iterated Integrals

In your multivariable calculus course you learned how to evaluate double and triple integrals as “iterated integrals” — that is, by integrating with respect to one variable at a time. Also, you learned that you could perform these “partial integrations” in any order. The Tonelli and Fubini Theorems are quite general justifications of those techniques for multiple integrals. We’ll start with a very general iterated integral, decomposing an integral in \( n + m \) variables into an integral in the first \( n \) variables followed by one in the last \( m \) variables. And we’ll show that this order can be reversed without changing the result.

Notation and Terminology 17.1. Throughout this section we identify \( \mathbb{R}^{n+m} \) with \( \mathbb{R}^n \times \mathbb{R}^m \). Also, for a function \( f \) on \( \mathbb{R}^{n+m} \) and \( y \in \mathbb{R}^m \) we write \( f(\cdot, y) \) for the function \( x \mapsto f(x, y) \), and similarly for \( f(x, \cdot) \).

If \( f \) is in \( L^+ \) or \( L^1 \) on \( \mathbb{R}^{n+m} \), then \( \int f(x,y) \, dx \) means regard \( f \) as a function of the first \( n \) coordinates \( x \), holding the last \( m \) coordinates \( y \) fixed, and integrate as a function of \( x \), and \( \iint f(x,y) \, dx \, dy \) means first evaluate the above integral \( \int f(x, y) \, dx \), getting a function of \( y \), then integrate this function; thus there are implied parentheses:

\[
\int \left( \int f(x, y) \, dx \right) \, dy.
\]

Theorem 17.2 (Tonelli’s Theorem). If \( f \in L^+(\mathbb{R}^{n+m}) \), then:

(i) \( f(\cdot, y) \) is measurable for almost all \( y \in \mathbb{R}^m \),
(ii) \( y \mapsto \int f(x, y) \, dx \) is measurable, and
(iii) \( \iint f(x, y) \, dx \, dy = \int f \).

Similarly for \( f(x, \cdot) \), \( x \mapsto \int f(x, y) \, dy \), and \( \iint f(x, y) \, dy \, dx \).

Proof. This is a hard one, requiring many steps.

Step 1. Let \( \mathcal{F} \) denote the set of functions in \( L^+ \) satisfying (i)–(iii). We start by establishing certain properties of \( \mathcal{F} \). It is completely routine to verify that if \( f \in \mathcal{F} \) then \( cf \in \mathcal{F} \) for every nonnegative real number \( c \). We’ll use the Monotone Convergence Theorem to show that \( \mathcal{F} \) is closed under series and increasing limits: for the first, let \( f_1, f_2, \ldots \in \mathcal{F} \) and put \( f = \sum_i f_i \). Then \( f(\cdot, y) = \sum_i f_i(\cdot, y) \) is measurable for almost all \( y \) since each \( f_i(\cdot, y) \) is measurable for almost all \( y \). By the Monotone Convergence Theorem \( \int f(x,y) \, dx = \sum_i \int f_i(x,y) \, dx \) for almost all \( y \).

\footnote{provided you took care with the “limits of integration”}
Again by the Monotone Convergence Theorem (twice),
\[
\iint f(x, y) \, dx \, dy = \sum_i \iint f_i(x, y) \, dx \, dy = \sum_i \int f_i = \int f.
\]
For increasing limits the argument is the same: if \(f_1, f_2, \cdots \in F\) and \(f_i \uparrow f\), then \(f(\cdot, y) = \lim_i f_i(\cdot, y)\) is measurable for almost all \(y\) since each \(f_i(\cdot, y)\) is measurable for almost all \(y\). By the Monotone Convergence Theorem \(\int f(x, y) \, dx = \lim_i \int f_i(x, y) \, dx\) for almost all \(y\), and
\[
\iint f(x, y) \, dx \, dy = \lim_i \iint f_i(x, y) \, dx \, dy = \lim_i \int f_i = \int f.
\]

**Step 2.** Thus it now suffices to show that \(F\) contains the characteristic function \(\chi_A\) of every measurable set \(A\), equivalently

(iii) \(A_y := \{x \mid (x, y) \in A\} \in M\) for almost all \(y\),

(v) \(y \mapsto m(A_y)\) is measurable, and

(vi) \(\int m(A_y) \, dy = m(A)\),

since
\[
\chi_A(\cdot, y) = \chi_{A_y},
\]
\[
\int \chi_A(x, y) \, dx = \int \chi_{A_y} = m(A_y),
\]
\[
\iint \chi_A(x, y) \, dx \, dy = \int m(A_y) \, dy, \quad \text{and}
\]
\[
\int \chi_A = m(A).
\]

Let \(C\) denote the family of measurable sets satisfying (iv)–(vi). We will show that \(C = M\) in a sequence of steps for different classes of \(A\). First note that the properties of \(F\) imply that \(C\) is closed under countable unions \(\bigcup_i A_i\) provided either \(A_1, A_2, \ldots\) are disjoint or \(A_1 \subset A_2 \subset \cdots\), since \(\chi_{\bigcup_i A_i}\) is the sum of the \(\chi_{A_i}\) in the first case and the increasing limit of the \(\chi_{A_i}\) in the second case.

**Step 3.** Let \(A\) be a box. Then \(A = B \times C\) for boxes \(B \subset \mathbb{R}^n\) and \(C \subset \mathbb{R}^m\). If \(y \in \mathbb{R}^m\) then
\[
A_y = \begin{cases} 
B & \text{if } y \in C \\
\emptyset & \text{if } y \notin C,
\end{cases}
\]
so \(A_y \in M\). Then \(m(A_y) = m(B)\chi_C(y)\), so \(y \mapsto m(A_y)\) is measurable, and
\[
\int m(A_y) \, dy = \int m(B)\chi_C = m(B)m(C) = m(A).
\]
Step 4. It now follows that \( \mathcal{C} \) contains every open set, because open sets are countable disjoint unions of boxes.

Step 5. Next we show that if \( A, B \in \mathcal{C} \) with \( A \subset B \) and \( m(A) < \infty \) (in particular, if \( A \) is bounded), then \( B \setminus A \in \mathcal{C} \): we have

\[
(B \setminus A)_y = B_y \setminus A_y,
\]

which is measurable for almost all \( y \). Since

\[
\infty > m(A) = \int m(A_y) \, dy,
\]

for almost all \( y \) we have \( m(A_y) < \infty \), hence \( m((B \setminus A)_y) = m(A_y) - m(A_y) \). Thus \( y \mapsto m((B \setminus A)_y) \) is measurable. Since \( y \mapsto m(A_y) \) is integrable, we have

\[
\int m((B \setminus A)_y) \, dy = \int (m(B_y) - m(A_y)) \, dy
= \int m(B_y) \, dy - \int m(A_y) \, dy
= m(B) - m(A) = m(B \setminus A).
\]

Step 6. Next we show that if \( A_1, A_2, \ldots \in \mathcal{C} \) with \( A_1 \) bounded and

\[
A_1 \supset A_2 \supset \cdots,
\]

then \( \bigcap_i A_i \in \mathcal{C} \), which will imply that \( \mathcal{C} \) contains all bounded \( G_\delta \) sets, since every bounded \( G_\delta \) set can be expressed as a countable decreasing intersection of bounded open sets. For each \( i \in \mathbb{N} \) put \( B_i = A_1 \setminus A_i \). Then \( \bigcup_i B_i \in \mathcal{C} \), being an increasing union of elements of \( \mathcal{C} \). Since \( \bigcup_i B_i \) is a bounded subset of \( A_1 \),

\[
\bigcap_i A_i = A_1 \setminus \bigcup_i B_i \in \mathcal{C}.
\]

Step 7. We now show that \( \mathcal{C} \) contains every bounded null set \( A \). Choose a bounded null \( G_\delta \) set \( B \ni A \). Then \( B \in \mathcal{C} \), so

\[
0 = m(B) = \int m(B_y) \, dy,
\]

hence \( m(B_y) = 0 \) for almost all \( y \). Since \( A_y \subset B_y \) for all \( y \in \mathbb{R}^m \), \( m(A_y) = 0 \) for almost all \( y \). Thus \( A_y \in \mathcal{M} \) for almost all \( y, y \mapsto m(A_y) \) is measurable, and

\[
\int m(A_y) \, dy = 0 = m(A).
\]
Step 8. Next we show that $\mathcal{C}$ contains every bounded measurable set $A$. Choose a bounded $G_\delta$ set $B \supset A$ such that $m(B \setminus A) = 0$. Then $B, B \setminus A \in \mathcal{C}$ and $B \setminus A$ is a bounded subset of $B$, so

$$A = B \setminus (B \setminus A) \in \mathcal{C}.$$ 

Step 9. Finally, it follows that $\mathcal{C} = \mathcal{M}$, because every measurable set is a countable increasing union of bounded measurable sets.

A similar argument proves the corresponding result with the roles of $x$ and $y$ reversed. \hfill \square

Exercise 17.3. Let $A$ and $B$ be measurable subsets of $\mathbb{R}$. Prove that $A \times B$ is measurable in $\mathbb{R}^2$, and $m(A \times B) = m(A)m(B)$.

Hint: the first part is not as trivial as it sounds; first of all, take the statements out of order by using Tonelli’s Theorem to show that if $A \times B$ is measurable then its measure is $m(A)m(B)$, then for the first part show that it suffices to consider $A \times \mathbb{R}$, and for this show that the family of all sets $A$ for which $A \times \mathbb{R}$ is measurable is a $\sigma$-algebra containing every Borel set and every null set. For the null sets, show that it’s enough to consider Borel null sets.

Exercise 17.4. Let $f$ be a measurable function on $\mathbb{R}$, and define $g$ on $\mathbb{R}^2$ by $g(x, y) = f(x)$. Prove that $g$ is measurable. Hint: use the definition of measurable function.

Exercise 17.5. Let $f : \mathbb{R} \to [0, \infty)$ be measurable, and put

$$G = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq f(x)\}.$$ 

Prove that $G$ is measurable and

$$m(G) = \int f.$$ 

Hint: for the first part, show that the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x, y) = f(x) - y$ is measurable. For the second part, use Tonelli’s Theorem to integrate with respect to $y$ first.

Theorem 17.6 (Fubini’s Theorem). Same as Tonelli, but $L^1$ instead of $L^+$.

Proof. Let $f \in L^1$. Then $f^+ \in L^1$, so by Tonelli’s Theorem

$$\int f^+ = \iint f^+(x, y) \, dx \, dy < \infty,$$

so

$$y \mapsto \int f^+(x, y) \, dx$$

is integrable,
hence $\int f^+(x, y) \, dx < \infty$ for almost all $y$. Thus $f^+(\cdot, y) \in L^1$ for almost all $y$, and similarly for $f^-$. Then

$$f(\cdot, y) = f^+(\cdot, y) - f^-(\cdot, y) \in L^1$$

for almost all $y$.

and then

$$y \mapsto \int f(x, y) \, dx = \int f^+(x, y) \, dx - \int f^-(x, y) \, dx$$

is integrable.

Then, by Tonelli’s Theorem again,

$$\int_{\mathbb{R}^2} \int f(x, y) \, dx \, dy = \int_{\mathbb{R}^2} \int f^+(x, y) \, dx \, dy - \int_{\mathbb{R}^2} \int f^-(x, y) \, dx \, dy$$

$$= \int f^+ - \int f^- = \int f. \quad \square$$

Here’s how these theorems are used to evaluate multiple integrals:

**Corollary 17.7.** If either $f \in L^+$ or $f \in L^1$, and if $(i_1, \ldots, i_n)$ is a rearrangement of $(1, \ldots, n)$, then

$$\int f = \int \cdots \int f(x_1, \ldots, x_n) \, dx_{i_1} \cdots dx_{i_n}.$$

**Proof.** Every permutation of $(1, \ldots, n)$ is a finite composition of transpositions of adjacent elements, and Tonelli-Fubini can be applied repeatedly, so without loss of generality

$$i_j = \begin{cases} 
  p & \text{if } j = p + 1 \\
  p + 1 & \text{if } j = p \\
  j & \text{if } j \neq p, p + 1
\end{cases}$$

for some $p = 1, \ldots, n - 1$. Applying Tonelli-Fubini twice, we have

$$\int f = \int_{\mathbb{R}^{n-p-1}} \int_{\mathbb{R}^{p-1}} \int_{\mathbb{R}^{p-1}} f(x_1, \ldots, x_n) \, dx_1 \cdots dx_{p-1} \, dx_{p+1} \cdots dx_n \, dx_p \, dx_{p+1}.$$

Again by Tonelli-Fubini, the order of integration with respect to $(x_p, x_{p+1})$ can be reversed, so we’re done. \quad \square

**Example 17.8.** Define $f$ on $A := (0, 1) \times (0, 1)$ by

$$f(x, y) = \frac{x - y}{(x + y)^3}.$$ 

We’ll show that

$$\int_0^1 \int_0^1 f(x, y) \, dy \, dx = \frac{1}{2} \quad \text{and} \quad \int_0^1 \int_0^1 f(x, y) \, dx \, dy = -\frac{1}{2}.$$

By Fubini’s Theorem, we can conclude that $f$ is not integrable on $A$.  

For the first iterated integral, we start with
\[
\int_{0}^{1} \frac{x - y}{(x + y)^3} \, dy.
\]
Note that this integral is infinite when \(x = 0\), but we can ignore this because the iterated integral can be regarded as
\[
\int_{(0,1)} \int_{(0,1)} x - y \frac{(x + y)^3}{(x + y)^3} \, dy \, dx,
\]
so without loss of generality we assume \(y > 0\).

We have
\[
\frac{x - y}{(x + y)^3} = \frac{2x - (x + y)}{(x + y)^3} = \frac{2x}{(x + y)^3} - \frac{1}{(x + y)^2},
\]
so
\[
\int_{0}^{1} \frac{x - y}{(x + y)^3} \, dy = \left[ -\frac{x}{(x + y)^2} + \frac{1}{x + y} \right]_{y=0}^{1} = \left[ \frac{y}{(x + y)^2} \right]_{y=0}^{1} = \frac{1}{(x + 1)^2}.
\]
Therefore
\[
\int_{0}^{1} \int_{0}^{1} \frac{x - y}{(x + y)^3} \, dy \, dx = \left[ -\frac{x + 1}{x + 1} \right]_{0}^{1} = \frac{1}{2}.
\]

For the other iterated integral, we can use symmetry:
\[
\int_{0}^{1} \int_{0}^{1} \frac{x - y}{(x + y)^3} \, dx \, dy = -\int_{0}^{1} \int_{0}^{1} \frac{y - x}{(y + x)^3} \, dx \, dy = -\frac{1}{2}.
\]

**Example 17.9.** Define \(f(x, y) = 1/x^3\) if \(0 < y < |x| < 1\) and 0 otherwise. Then \(\int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y) \, dx \, dy = 0\) and \(\int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y) \, dy \, dx\) does not exist.

**Exercise 17.10.** Define \(f(x, y) = xy/(x^2 + y^2)^2\) away from the origin, and let \(f(0, 0) = 0\). Prove:
(a) \(\int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y) \, dx \, dy = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y) \, dy \, dx = 0\), but
(b) \(f\) is not integrable.

**Exercise 17.11.** Let \(f\) be integrable on \((0, a)\) (meaning that \(f\) is measurable on \((0, a)\), and the trivial extension of \(f\) to \(\mathbb{R}\), with constant value 0 outside \((0, a)\), is integrable). Prove that
\[
\int_{0}^{a} \int_{y}^{a} f(x) \frac{x}{y} \, dx \, dy = \int_{0}^{a} f.
\]
Hint: use Tonelli’s Theorem to prove that the function \((x, y) \mapsto f(x)/x\) is integrable on the set
\[
A := \{(x, y) \mid 0 \leq y \leq a, y \leq x \leq a\},
\]
then use Fubini’s Theorem to interchange the order of integration.
Example 17.12. Although Lebesgue integrals subsume Riemann integrals, they do do not subsume improper integrals. For example, you probably learned in a previous analysis course that the improper integral
\[
\int_0^\infty \left| \frac{\sin x}{x} \right| \, dx
\]
diverges, but the improper integral
\[
\int_0^\infty \frac{\sin x}{x} \, dx
\]
converges. In the language of Lebesgue integrals, the function \( x \mapsto \frac{\sin x}{x} \) is not integrable on \((0, \infty)\), although the limit
\[
\lim_{b \to \infty} \int_0^b \frac{\sin x}{x} \, dx
\]
exists. In the next exercise you'll use Tonelli-Fubini to compute the value of this limit.

Exercise 17.13. In this exercise you'll show that
\[
\lim_{b \to \infty} \int_0^b \frac{\sin x}{x} \, dx = \frac{\pi}{2}.
\]
Let \( f(x, y) = e^{-xy} \sin x \) and \( A = (0, b) \times (0, \infty) \) (where without loss of generality \( b > 0 \)).

(a) Prove that \( f \) is integrable on \( A \) by carefully estimating \(|f|\).
(b) Integrate \( f \) over \( A \) in one order to get \( \int_0^b \sin x/x \, dx \).
(c) Integrate \( f \) over \( A \) in the other order to get
\[
\frac{\pi}{2} - \sin b \int_0^\infty \frac{ye^{-by}}{1+y^2} \, dy - \cos b \int_0^\infty \frac{e^{-by}}{1+y^2} \, dy.
\]
(d) Show how the Dominated Convergence Theorem can be used to help prove that both of the above integrals go to 0 as \( b \to \infty \).
(e) Indicate how this implies the desired result.

Exercise 17.14. Let \( 0 < a < b < \infty \). Integrate \( e^{-xy} \) over \( A := (0, \infty) \times (a, b) \) in two different ways to help show that
\[
\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \, dx = \log \frac{b}{a}.
\]