Exercise 0.1. (Translation-Invariance of Lebesgue Integral) Let $c \in \mathbb{R}^n$. Prove that if $f$ is in $L^+$ or $L^1$, then so is $x \mapsto f(x+c)$, and
\[
\int f(x+c)\,dx = \int f.
\]
Hint: first show it for simple functions using linearity of the integral and translation-invariance of Lebesgue measure. Then show it for $L^+$ using the Monotone Convergence Theorem. Then finally show it for $L^1$ using positive and negative parts.

Exercise 0.2. (Measurability of Derivatives) Let $U \subset \mathbb{R}$ be open and $f: U \rightarrow \mathbb{R}$ be differentiable. Prove that $f'$ is measurable.

Exercise 0.3. (Continuity of Translation) Let $f \in L^1$, and define $g: \mathbb{R}^n \rightarrow L^1$ by
\[
g(t)(x) = f(x+t).
\]
Prove that $g$ is continuous. Hint: first do it for a continuous function with compact support.

Exercise 0.4. (Riemann-Lebesgue Lemma) Let $f \in L^1(\mathbb{R})$, and define $g: \mathbb{R} \rightarrow \mathbb{R}$ by
\[
g(t) = \int f(x) \cos xt\,dx.
\]
Prove that
\[
\lim_{t \rightarrow \infty} g(t) = 0.
\]
Hint: first do it for a characteristic function of a bounded interval.

Exercise 0.5. (Absolute Continuity) Let $f \in L^1$. Prove that for all $\epsilon > 0$ there exists $\delta > 0$ such that
\[
\left| \int_A f \right| < \epsilon \quad \text{if } m(A) < \delta.
\]
Hint: consider a sequence $(A_i)$ in $\mathcal{M}$ with $m(A_i) \rightarrow 0$. 

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Exercise 0.6. Let $f \in L^1$, and let $A_1, A_2, \ldots \in \mathcal{M}$ be disjoint. Prove that
\[
\int_{\bigcup_{i=1}^{\infty} A_i} f = \sum_{i=1}^{\infty} \int_{A_i} f.
\]

Exercise 0.7. Let $f \in L^1(\mathbb{R})$, and suppose $\int_{0}^{t} f = 0$ for all $t \in \mathbb{R}$. Prove that $f = 0$ almost everywhere. Hint: sets of finite measure can be approximated by finite unions of intervals.