MAT 473 Homework 9
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Due: Thursday, March 30

1. For each $f \in L^1(\mathbb{R})$ define $\Psi(f) : \mathbb{R} \to \mathbb{R}$ by
   \[ \Psi(f)(t) = \int_{-\infty}^{\infty} f(x) \sin(xt) \, dx. \]
   
   (a) Prove that if $f_n \to f$ in $L^1$ then $\Psi(f_n) \to \Psi(f)$ uniformly on $\mathbb{R}$.
   
   (b) Prove that for every $f \in L^1(\mathbb{R})$,
   \[ \lim_{t \to \infty} \Psi(f)(t) = 0. \]
   
   Hint: first take $f$ to be the characteristic function of a bounded interval. You may use simple change-of-variables rules for integrals.

2. Find a reference somewhere (any good book on real analysis should do) that talks about “fat Cantor sets”, that is, nowhere dense compact subsets of $\mathbb{R}$ with positive measure, and briefly describe the construction (you need not strive for maximum generality — it is enough to have at least one such set). If $A$ is a fat Cantor set, is the characteristic function of $A$ Riemann integrable? Why or why not?

3. Give examples on $[0,1]$ (and prove that they have the stated properties) of each of the following phenomena:
   
   (a) a function $f$ and a continuous function $g$ such that $f = g$ a.e. and $f$ is nowhere continuous;
   
   (b) a function $p$ which is continuous a.e. such that there is no continuous function $q$ with $p = q$ a.e.