1. Let $f$ and $g$ be continuous integrable functions on $\mathbb{R}$. Prove that the formula

$$f * g(x) := \int_{-\infty}^{\infty} f(x - y)g(y) \, dy$$

makes sense for all $x \in \mathbb{R}$, and use Tonelli’s Theorem, together with a simple change of variables, to prove that the resulting function $f * g$ is an element of $L^1$.

Note: we only impose the continuity hypothesis to ease the measurability considerations.

2. Let $f : \mathbb{R} \to [0, \infty)$ be measurable, and put

$$G = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq f(x)\}.$$

Prove that $G$ is measurable and

$$m(G) = \int f.$$

3. Show that

$$\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} \, dy \, dx = \frac{1}{2} \quad \text{and} \quad \int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} \, dx \, dy = -\frac{1}{2}.$$

What can you conclude about the function $(x, y) \mapsto (x - y)/(x + y)^3$?