MAT 473 HOMEWORK 6

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1. Let $A$ be a measurable subset of $\mathbb{R}^p$, and let $t < m(A)$. Prove that there exists a compact set $K$ such that $K \subseteq A$ and $m(K) > t$. However, show that, given $\epsilon > 0$, we cannot expect to be able to find a compact set $K$ such that $m(A \setminus K) < \epsilon$.

2. Let $A \subseteq \mathbb{R}^p$, and suppose that for all $\epsilon > 0$ there exist an open set $B$ and a closed set $C$ such that $C \subseteq A \subseteq B$ and $m(B \setminus C) < \epsilon$. Prove that $A$ is measurable.

3. Define an equivalence relation $\sim$ on $\mathbb{R}$ by

$$x \sim y \quad \text{if and only if} \quad x - y \in \mathbb{Q}.$$ 

By the Axiom of Choice, there exists a set $A$ consisting of exactly one element of each equivalence class (similarly to Homework 5, but now the equivalence relation is on all of $\mathbb{R}$ rather than just $[0, 1]$). Then $\mathbb{R} = \bigcup_{x \in \mathbb{Q}} (A + x)$, and for distinct $x, y \in \mathbb{Q}$ we have $(A + x) \cap (A + y) = \emptyset$, again similarly to Homework 5. You do not have to prove any of this so far.

(a) Prove that if $E \subseteq A$ and $E$ is measurable, then $m(E) = 0$. Hint: Suppose not. Then there is a compact subset $K$ of $E$ of positive measure. Put $C = \mathbb{Q} \cap [0, 1]$. Deduce that $\bigcup_{x \in C} (K + x)$ is a bounded measurable set with infinite measure.

(b) Prove that if $B$ is a measurable subset of $\mathbb{R}$ with $m(B) > 0$, then $B$ has a nonmeasurable subset. Hint:

$$B = \bigcup_{x \in \mathbb{Q}} (B \cap (A + x)),$$

where $A$ is the same as above.

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