MAT 473 HOMEWORK 3

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1. Let $U \subseteq \mathbb{R}^p$ be open, and let $f : U \to \mathbb{R}^q$. Suppose every partial derivative $D_j f_i$ is bounded on $U$. Prove that $f$ is continuous on $U$. Hint: the Mean Value Inequality does not apply. Use the technique of the proof that if every $D_j f_i$ is continuous then $f$ is differentiable.

2. Prove that there exist $r > 0$ and continuously differentiable real-valued functions $u, v, w$ defined on the open ball $B_r(1, 1)$ in $\mathbb{R}^2$ such that $u(1, 1) = 1$, $v(1, 1) = 1$, $w(1, 1) = -1$, and for all $(x, y) \in B_r(1, 1),
\begin{align*}
u^5 + xv^2 - y + w &= 0 \\
v^5 + yu^2 - x + w &= 0 \\
w^4 + y^5 - x^4 &= 1.
\end{align*}

Also, find $D_2 v(1, 1)$.

3. Define $f : \mathbb{R} \to \mathbb{R}$ by
\[ f(x) = \begin{cases} 
  x + 2x^2 \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\
  0 & \text{if } x = 0.
\end{cases} \]

Show that $f$ is differentiable on $\mathbb{R}$ and $f'(0) \neq 0$, but $f$ is not 1-1 on any open interval containing 0. Why does this not contradict the Inverse Function Theorem?

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