Throughout, $X$ will denote a metric space. For convenience, we assume the equivalence of compactness and sequential compactness.

**Definition.** $X$ is *totally bounded* if for all $r > 0$ there exists a finite subset $F \subset X$ such that $X = \bigcup_{t \in F} B_r(t)$.

**Theorem 0.1.** $X$ is compact if and only if it is complete and totally bounded.

**Proof.** In our earlier proofs we have already seen that compactness implies completeness and total boundedness. Conversely, assume that $X$ is complete and totally bounded, and let $(x_n)$ be a sequence in $X$. We will show that $(x_n)$ has a convergent subsequence, thus establishing sequential compactness, which as we know is equivalent to compactness.

First choose a finite subset $F_1 \subset X$ such that $X = \bigcup_{t \in F_1} B_1(t)$.

Then for some $t \in F_1$ the set $\{n \in \mathbb{N} : x_n \in B_1(t)\}$ must be infinite. Thus, there is a subsequence $(x_{1,n})$ of $(x_n)$ which is contained in a ball of radius 1. Similarly, there is a subsequence $(x_{2,n})$ of $(x_{1,n})$ which is contained in a ball of radius $1/2$. Continue in this way, getting a sequence of successive subsequences of $(x_n)$, more precisely, for each $k = 2, 3, \ldots$ we have a subsequence $(x_{k,n})$ of $(x_{k-1,n})$ which is contained in a ball of radius $1/k$.

We use the Cantor diagonalization trick: define a subsequence $(y_k)$ of $(x_n)$ by

$$y_k = x_{k,k}.$$ 

Then for each $k \in \mathbb{N}$ the tail $(y_k, y_{k+1}, y_{k+2}, \ldots)$ of $(y_k)$ is contained in a ball of radius $1/k$. Thus

$$d(y_k, y_j) < \frac{1}{2k} \quad \text{for all } j \geq k.$$ 

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It follows that the subsequence \((y_k)\) is Cauchy, hence convergent since \(X\) is complete. \(\square\)