**RATIOS AND ROOTS**

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**Proposition 0.1.** Let \((a_n)\) be a sequence of positive numbers. Then

\[
\lim \inf \frac{a_{n+1}}{a_n} \leq \lim \inf a_n^{1/n}.
\]

**Proof.** Since \(a_n > 0\) for all \(n\), both \(\lim\) infs are nonnegative. The inequality is trivial if the left-hand side is 0, so assume \(\lim \inf \frac{a_{n+1}}{a_n} > 0\). Let

\[
0 < t < \lim \inf \frac{a_{n+1}}{a_n}.
\]

Deleting finitely many terms of the sequence \((a_n)\), without loss of generality

\[
\frac{a_{n+1}}{a_n} > t \quad \text{for all } n \in \mathbb{N}.
\]

Thus

\[
a_2 \geq ta_1
\]

\[
a_3 \geq ta_2 \geq t^2a_1
\]

\[
\ldots
\]

\[
a_n \geq t^{n-1}a_1 \quad \text{for all } n \geq 2.
\]

Taking \(n\)th roots:

\[
a_n^{1/n} \geq t^{(n-1)/n}a_1^{1/n} \quad \text{for all } n \geq 2.
\]

Now, \(t^{(n-1)/n} \to t\) because \((n-1)/n \to 1\), and \(a_1^{1/n} \to 1\) because \(a_1 > 0\). Thus letting \(n \to \infty\) we get

\[
\lim \inf a_n^{1/n} \geq t.
\]

Since \(t\) was arbitrary in the interval \((0, \lim \inf \frac{a_{n+1}}{a_n})\), the desired inequality follows.

\[\square\]

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