Integrals and Composition

**Proposition 1.** Let \( f : [a, b] \to [c, d] \) be integrable, and let \( g \) be uniformly continuous on \([c, d]\). Then \( g \circ f \) is integrable.

**Proof.** Let \( \varepsilon > 0 \). Choose \( \delta > 0 \) such that for all \( u, v \in [c, d] \), if \(|u - v| < \delta\) then
\[
|g(u) - g(v)| < \frac{\varepsilon}{2(b - a)}.
\]

Choose \( M > 0 \) such that \( |g| \leq M \) on \([c, d]\). Choose step functions \( h, k \) such that
\[
c \leq h \leq f \leq k \leq d \quad \text{and} \quad \int_a^b (k - h) < \frac{\varepsilon \delta}{4M}.
\]

Choose \( c_1, \ldots, c_n, d_1, \ldots, d_n \in \mathbb{R} \) and pairwise disjoint intervals \( A_1, \ldots, A_n \) such that
\[
h = \sum_{i=1}^n c_i \chi_{A_i},
\]
\[
k = \sum_{i=1}^n d_i \chi_{A_i}.
\]

Note that
\[
\frac{\varepsilon \delta}{4M} > \int_a^b (k - h) = \sum_{i=1}^n (d_i - c_i) \ell(A_i) \geq \sum_{d_i - c_i \geq \delta} \delta \ell(A_i),
\]
so
\[
\sum_{d_i - c_i \geq \delta} \ell(A_i) < \frac{\varepsilon}{4M}.
\]

Then \( g \circ h \) and \( g \circ k \) are step functions with
\[
g \circ h \leq g \circ f \leq g \circ k,
\]
and
\[
\int_a^b (g \circ k - g \circ h) = \sum_{i=1}^n (g(d_i) - g(c_i)) \ell(A_i)
\]
\[
= \sum_{d_i - c_i \geq \delta} (g(d_i) - g(c_i)) \ell(A_i) + \sum_{d_i - c_i < \delta} (g(d_i) - g(c_i)) \ell(A_i)
\]
\[
< 2M \sum_{d_i - c_i \geq \delta} \ell(A_i) + \frac{\varepsilon}{2(b - a)} \sum_{d_i - c_i < \delta} \ell(A_i)
\]
\[
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qed
\]

**Corollary 2.** Every continuous function on \([a, b]\) is integrable.
Proof. Since every continuous function on \([a, b]\) is uniformly continuous, and since the function \(x \mapsto x\) is integrable because it is monotone, the result follows from Proposition 1. \(\square\)

**Corollary 3.** If \(f\) is integrable on \([a, b]\), then so is \(|f|\), and

\[
\left| \int_a^b f \right| \leq \int_a^b |f|.
\]

Proof. Since \(x \mapsto |x|\) is uniformly continuous on the range of \(f\), the first part follows immediately from Proposition 1, and for the other part just integrate the inequality

\[-|f| \leq f \leq |f|.
\]

\(\square\)

**Corollary 4.** If \(f\) and \(g\) are integrable on \([a, b]\), then so is \(fg\).

Proof. It suffices to show that \(f^2\) is integrable, since

\[
fg = \frac{(f + g)^2 - (f - g)^2}{4}.
\]

The function \(x \mapsto x^2\) is uniformly continuous on the range of \(f\), so the result follows from Proposition 1. \(\square\)