Here we give a new proof of another theorem in the lecture notes:

**Theorem 1** (Arzela-Ascoli Theorem). Let $X$ be a compact metric space and $A \subset C(X)$. Then $\overline{A}$ is compact if and only if $A$ is pointwise bounded and equicontinuous.

**Proof.** We use the following: $A$ is totally bounded if and only if $\overline{A}$ is, consequently $\overline{A}$ is compact if and only if $A$ is totally bounded. Thus it suffices to show that $A$ is totally bounded if and only if it is pointwise bounded and equicontinuous. The forward direction is adequately proved in the lecture notes (although $\overline{A}$ can be replaced by $A$ there).

We give a new proof of the converse, however: assume $A$ is pointwise bounded and equicontinuous. We must show that $A$ is totally bounded. Given $\varepsilon > 0$, choose $\delta > 0$ such that $d(x, y) < \delta$ implies $|f(x) - f(y)| < \varepsilon/3$ for all $f \in A$. Since $X$ is compact, it is totally bounded, so we can choose $x_1, \ldots, x_n \in X$ such that $X = \bigcup_{i=1}^{n} B_\delta(x_i)$. Define $q : A \rightarrow \mathbb{R}^n$ by

$$q(f) = (f(x_1), \ldots, f(x_n)).$$

Then $q(A)$ is bounded, hence totally bounded by the Heine-Borel Theorem. Thus there is a finite $S \subset A$ such that

$$q(A) \subset \bigcup_{g \in S} B_{\varepsilon/3}(q(g)).$$

Let $f \in A$. Choose $g \in S$ such that $\|q(f) - q(g)\| < \varepsilon/3$, so that

$$|f(x_i) - g(x_i)| < \frac{\varepsilon}{3} \quad \text{for all } i = 1, \ldots, n.$$

Let $x \in X$. Choose $i$ such that $d(x, x_i) < \delta$. Then

$$|f(x) - g(x)| \leq |f(x) - f(x_i)| + |f(x_i) - g(x_i)| + |g(x_i) - g(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Since $|f - g|$ is continuous and $X$ is compact, we get $\|f - g\| < \varepsilon$, and we have shown that $A$ is totally bounded.