Integration

In this section we derive the basic facts concerning the Riemann integral, but in a somewhat nonstandard way. ¹

Standing Hypothesis. Throughout this section, all our functions \( f \) will have domain and range in \( \mathbb{R} \) unless otherwise specified. Also, \( [a,b] \) will be a compact interval (possibly of zero length). Furthermore, throughout this section all intervals will be bounded unless otherwise specified.

Definition 1. The characteristic function of a subset \( A \subset \mathbb{R} \) is defined by
\[
\chi_A(x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{if } x \notin A.
\end{cases}
\]

Definition 2. A function is supported in \( A \) if it vanishes (i.e., is zero) outside of \( A \).

Definition 3. A step function is a linear combination of characteristic functions of intervals.

Observation 4. Let \( g \) and \( h \) be step functions.

1. The following are also step functions:
   - \( cg \) for \( c \in \mathbb{R} \);
   - \( g + h \);
   - \( gh \);
   - \( \max\{g,h\} \);
   - \( \min\{g,h\} \).

2. There exists \( c_1, \ldots, c_n, d_1, \ldots, d_n \in \mathbb{R} \) and pairwise disjoint intervals \( A_1, \ldots, A_n \) such that
   \[
   g = \sum_{i=1}^{n} c_i \chi_{A_i} \quad \text{and} \quad h = \sum_{i=1}^{n} d_i \chi_{A_i}.
   \]

Definition 5. The length of an interval \( A \) is denoted by \( \ell(A) \). Thus,
\[
\ell(a,b) = \ell[a,b] = \ell[a,b] = \ell(a,b) = b - a.
\]

Observation 6. If \( A \) is an interval and \( B_1, \ldots, B_k \) are pairwise disjoint intervals such that \( A \subset \bigcup_{j=1}^{k} B_j \), then
\[
\ell(A) = \sum_{j=1}^{k} \ell(A \cap B_j).
\]

¹You should really be learning the Lebesgue integral, because it’s much more powerful. On the other hand the Riemann integral in one real variable is much easier to develop, and you need to be familiar with it anyway, because it appears so often in mathematical literature. And besides, here we’ll rigorously justify all the results about integration you saw in your calculus course.
Definition 7. Let \( g = \sum_{i=1}^{n} c_i \chi_{A_i} \) be a step function, where \( A_1, \ldots, A_n \) are pairwise disjoint intervals. The integral of \( g \) is
\[
\int g := \sum_{i=1}^{n} c_i \ell(A_i).
\]

Lemma 8. If \( g \) is a step function, then \( \int g \) is well-defined.

Proof. Let
\[
g = \sum_{i=1}^{n} c_i \chi_{A_i} = \sum_{j=1}^{k} d_j \chi_{B_j}
\]
with both families \( \{A_1, \ldots, A_n\} \) and \( \{B_1, \ldots, B_k\} \) of intervals pairwise disjoint. If necessary, add terms with zero coefficients so that \( \bigcup_{i=1}^{n} A_i = \bigcup_{j=1}^{k} B_j \). Note each \( A_i \cap B_j \) is an interval, and if it is nonempty then \( c_i = d_j \). We have
\[
\sum_{i=1}^{n} c_i \ell(A_i) = \sum_{i=1}^{n} c_i \sum_{j=1}^{k} \ell(A_i \cap B_j)
= \sum_{i=1}^{n} \sum_{j=1}^{k} c_i \ell(A_i \cap B_j)
= \sum_{j=1}^{k} \sum_{i=1}^{n} \ell(A_i \cap B_j)
= \sum_{j=1}^{k} \sum_{i=1}^{n} \ell(A_i \cap B_j)
= \sum_{j=1}^{k} d_j \ell(B_j).
\]

Proposition 9. For all step functions \( g \) and \( h \),
\[
\begin{align*}
(1) \quad & \int g \leq \int h; \\
(2) \quad & \int cg = c \int g \text{ if } c \in \mathbb{R}; \\
(3) \quad & \int (g + h) = \int g + \int h.
\end{align*}
\]
Proof. (1)–(3) are obvious if we write \( g = \sum_{i=1}^{n} c_i \chi_{A_i} \) and \( h = \sum_{i=1}^{n} d_i \chi_{A_i} \) for pairwise disjoint intervals \( A_1, \ldots, A_n \).

Definition 10. A bounded function \( f \) on \([a, b]\) is integrable if
\[
\sup \left\{ \int g : g \text{ step function}, g \leq f \right\} = \inf \left\{ \int h : h \text{ step function}, f \leq h \right\},
\]
in which case this common value is the integral of \( f \) from \( a \) to \( b \), denoted \( \int_{a}^{b} f \) or \( \int_{a}^{b} f(x) \, dx \). \( f \) is the integrand, \( a \) is the lower limit of integration, and \( b \) is the upper limit of integration. If \([a, b] \subset \text{dom} \, f\), we say \( f \) is integrable on \([a, b]\) if \( f|_{[a,b]} \) is integrable.
Observation 11.  (1) A bounded function $f$ on $[a,b]$ is integrable if and only if for all $\varepsilon > 0$ there exist step functions $g, h$ such that $g \leq f \leq h$ and $\int (h - g) < \varepsilon$.

(2) If $f : [a, b] \to \mathbb{R}$ and $[c, d] \subset [a, b]$, then $f$ is integrable on $[c, d]$ if and only if $f \chi_{[c,d]}$ is integrable on $[a, b]$, in which case

$$\int_c^d f = \int_a^b f \chi_{[c,d]}.$$

Proposition 12 (Monotonicity of Integrals). Let $f$ and $g$ be integrable on $[a, b]$. If $f \leq g$ then $\int_a^b f \leq \int_a^b g$.

Proof. Let $h$ be a step function with $h \leq f$. Then $h \leq g$, so $\int h \leq \int_a^b g$. Taking the sup over $h$, we get $\int_a^b f \leq \int_a^b g$. □

Proposition 13 (Arithmetic of Integrals). If $f$ and $g$ are integrable on $[a, b]$, then:

1. $\int_a^b (f + g) = \int_a^b f + \int_a^b g$;
2. $\int_a^b cf = c \int_a^b f$ if $c \in \mathbb{R}$.

Proof. (1) Given $\varepsilon > 0$, choose step functions $h, k, l, m$ such that $h \leq f \leq k$, $l \leq g \leq m$, $\int (k - h) < \frac{\varepsilon}{2}$, and $\int (m - l) < \frac{\varepsilon}{2}$.

Then $h + l$ and $k + m$ are step functions,

$$h + l \leq f + g \leq k + m,$$

and

$$\int ((k + m) - (h + l)) = \int (k - h) + \int (m - l) < \varepsilon.$$

Thus $f + g$ is integrable.

For the other part, we have

$$\int (h + l) = \int h + \int l \leq \int_a^b f + \int_a^b g \leq \int k + \int m = \int (k + m),$$

and

$$\int (h + l) \leq \int_a^b (f + g) \leq \int (k + m),$$

so the numbers $\int_a^b f + \int_a^b g$ and $\int_a^b (f + g)$ are closer together than $\varepsilon$, hence are equal since $\varepsilon > 0$ was arbitrary.

(2) First of all, the result is trivial if $c = 0$. Next consider the case $c > 0$. Given $\varepsilon > 0$, choose step functions $h, k$ such that $h \leq f \leq k$ and $\int (k - h) < \varepsilon/c$. Then $ch$ and $ck$ are step functions, $ch \leq cf \leq ck$, and

$$\int (ck - ch) = c \int (k - h) < \varepsilon.$$

Thus $cf$ is integrable. Now just note that the two numbers $\int_a^b cf$ and $c \int_a^b f$ are between $c \int h$ and $c \int k$, hence are closer together than $\varepsilon$, so are equal since $\varepsilon$ was arbitrary.

The case $c < 0$ is similar, except that now we have $ck \leq cf \leq ch$ and $\int (ch - ck) < \varepsilon$. □
Theorem 14. Every continuous function on \([a, b]\) is integrable.

Proof. \(f\) is uniformly continuous since \([a, b]\) is compact, so, given \(\varepsilon > 0\) we can choose \(\delta > 0\) such that
\[
|f(x) - f(y)| < \frac{\varepsilon}{2(b-a)} \quad \text{whenever } |x - y| < \delta.
\]
Thus we can partition \([a, b]\) into subintervals \(A_1, \ldots, A_n\) such that
\[
\sup_{A_i} f - \inf_{A_i} f < \frac{\varepsilon}{b-a} \quad \text{for } i = 1, \ldots, n,
\]
so we can find step functions \(g, h\) such that \(g \leq f \leq h\) and \(h - g < \varepsilon/(b-a)\), hence \(\int (h - g) < \varepsilon\). \(\square\)

Theorem 15. Every monotone function on \([a, b]\) is integrable.

Proof. Multiplying by \(-1\) if necessary, we can assume that \(f\) is increasing. Given \(\varepsilon > 0\), choose \(n \in \mathbb{N}\) such that \((f(b) - f(a))(b - a)/n < \varepsilon\). Divide \([a, b]\) into \(n\) nonoverlapping intervals \([a_i, b_i]\) of equal length \((b - a)/n\), numbered so that the \(a_i\) and \(b_i\) are increasing. Put
\[
g = \sum_{i=1}^{n} f(a_i) \chi_{[a_i, b_i]} + f(b) \chi_{b} \\
h = \sum_{i=1}^{n} f(b_i) \chi_{[a_i, b_i]} + f(b) \chi_{b}.
\]
Then \(g \leq f \leq h\) and
\[
\int (h - g) = \sum_{i=1}^{n} (f(b_i) - f(a_i)) \frac{b-a}{n} = \frac{(f(b) - f(a))(b - a)}{n} < \varepsilon.
\]
\(\square\)

Definition 16. For \(x \in \mathbb{R}\), the positive and negative parts of \(x\) are
\[x^+ := \max\{x, 0\} \quad \text{and} \quad x^- := \max\{-x, 0\}.
\]

Observation 17. With the above notation, we have:

1. (a) \(x^+, x^- \geq 0\);
   (b) \(x^- = (-x)^+\);
   (c) \(x = x^+ - x^-\);
   (d) \(|x| = x^+ + x^-\);
   (e) \(x^+x^- = 0\);
   (f) \((cx)^+ = \begin{cases} cx^+ & \text{if } c \geq 0 \\ cx^- & \text{if } c < 0 \end{cases}\)
   \(\text{and } (cx)^- = \begin{cases} cx^- & \text{if } c \geq 0 \\ -cx^+ & \text{if } c < 0 \end{cases}\);
   (g) \((x + y)^+ \leq x^+ + y^+\);
   (h) \(x^+ - y^+ \leq (x - y)^+\).

2. The function \(x \mapsto x^+\) is increasing and continuous.

Definition 18. Let \(f : A \to \mathbb{R}\). The positive and negative parts of \(f\) are the functions \(f^+, f^- : A \to \mathbb{R}\) defined by
\[f^+(x) = f(x)^+ \quad \text{and} \quad f^-(x) = f(x)^-.
\]
Proposition 19. If $f$ and $g$ are integrable on $[a, b]$, then so is $fg$.

Proof. First note that it suffices to show that $f^2$ is integrable, because

$$fg = \frac{(f + g)^2 - (f - g)^2}{4}.$$ 

Also, it suffices to prove it for the case $f \geq 0$, since

$$f^2 = (f^+)^2 - (f^-)^2.$$ 

Choose $c \in (0, \infty)$ such that $f < c$. Given $\varepsilon > 0$, choose step functions $h, k$ such that

$$0 \leq h \leq f \leq k \leq c \quad \text{and} \quad \int (k - h) < \frac{\varepsilon}{2c}.$$ 

Then $h^2$ and $k^2$ are step functions, $h^2 \leq f^2 \leq k^2$, and

$$k^2 - h^2 = (k + h)(k - h) \leq 2c(k - h),$$

so

$$\int (k^2 - h^2) \leq 2c \int (k - h) < \varepsilon. \quad \square$$

Corollary 20. If $f$ is integrable on $[a, b]$, then $f$ is integrable on $[c, d]$ for all $[c, d] \subset [a, b]$.

Proposition 21. If $f$ is integrable on $[a, b]$, then

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$ 

Proof. First we show that $f^+$ is integrable: given $\varepsilon > 0$, choose step functions $g, h$ such that $g \leq f \leq h$ and $\int (h - g) < \varepsilon$. Then $g^+$ and $h^+$ are step functions, $g^+ \leq f^+ \leq h^+$, and

$$\int (h^+ - g^+) \leq \int (h - g) < \varepsilon.$$ 

Then $f^+$ is integrable, hence so is $|f| = f^+ - f^-$. For the other part, we have $-|f| \leq f \leq |f|$, so

$$-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|,$$

which implies the desired inequality. \quad \square

Definition 22. Define

$$\int_a^b f = \begin{cases} -\int_b^a f & \text{if } a > b \\ 0 & \text{if } a = b. \end{cases}$$

Observation 23. Let $f$ be integrable on a closed interval $I$. Then

$$\int_a^c f = \int_a^b f + \int_b^c f \quad \text{for all } a, b, c \in I,$$

as can be seen by considering the cases determined by the possible relative positions of $a, b, c$.

Most of our results on integrals have true analogues if the limits of integration are equal or backwards. However, when we write $\int_a^b f$ without specifying the ordering of $a, b$, by default we intend that $a < b$. 
Theorem 24. Let $I$ be an interval, let $f : I \to \mathbb{R}$ be bounded, and let $a \in I$. Suppose that $\int_a^x f$ exists for all $x \in I$. Then the function $x \mapsto \int_a^x f$ is continuous on $I$.

Proof. Put $M = \sup |f|$. Given $\varepsilon > 0$, choose $\delta > 0$ such that $M\delta < \varepsilon$. Then for all $x, y \in I$, if $y \leq x < y + \delta$ we have
\[
\left| \int_a^x f - \int_a^y f \right| = \left| \int_y^x f \right| \leq \int_y^x |f| \leq M(x - y) < \varepsilon.
\]
\[\square\]

Corollary 25. Let $f : [a, b] \to \mathbb{R}$ be bounded, and suppose $f$ is integrable on $[a, t]$ for every $t \in (a, b)$. Then $f$ is integrable on $[a, b]$, and
\[
\int_a^b f = \lim_{t \to b} \int_a^t f.
\]
Similarly if $f$ is integrable on $[t, b]$ for every $t \in (a, b)$.

Proof. We only prove the first statement; the second then follows from a similar argument. Choose $c, d \in \mathbb{R}$ such that $c < f < d$. Given $\varepsilon > 0$, choose $t \in (a, b)$ such that $(d - c)(b - t) < \varepsilon/2$. Since $f$ is integrable on $[a, t]$ by hypothesis, we can choose step functions $g, h$ supported on $[a, t]$ such that $g \leq f \leq h$ on $[a, t]$ and $\int (h - g) < \varepsilon/2$. Put
\[g' = g + c\chi_{(t, b]} \quad \text{and} \quad h' = h + d\chi_{(t, b]}.
\]
Then $g'$ and $h'$ are step functions, $g' \leq f \leq h'$ on $[a, b]$, and
\[
\int (h' - g') = \int (h - g) + (d - c)(b - t) < \varepsilon.
\]
Thus $f$ is integrable on $[a, b]$, and then $\int_a^t f \to \int_a^b f$ by Theorem 24. \[\square\]

Example. The function taking values $\sin(1/x)$ for $x \neq 0$ and 0 at 0 is integrable\(^2\) on $[0, 1]$.

Corollary 26. Let $f, g : [a, b] \to \mathbb{R}$. If $f$ is integrable and $f = g$ except $a$ or $b$, then $g$ is integrable and $\int_a^b f = \int_a^b g$.

Theorem 27 (Interval Additivity). Let $a < b < c$ and $f : [a, c] \to \mathbb{R}$. If $f$ is integrable on both $[a, b]$ and $[b, c]$, then $f$ is integrable on $[a, c]$ and
\[
\int_a^c f = \int_a^b f + \int_b^c f.
\]

Proof. Define $g, h : [a, c] \to \mathbb{R}$ by
\[g = f\chi_{[a, b]} \quad \text{and} \quad h = f\chi_{(b, c]}.
\]
Then $g$ is integrable on $[a, c]$ and $\int_a^c g = \int_a^b f$. Since $h = f\chi_{[b, c]}$ except possibly at $b$, $h$ is integrable on $[a, c]$ and $\int_a^c h = \int_b^c f$. We have $f = g + h$, so $f$ is integrable on $[a, c]$ and
\[
\int_a^c f = \int_a^c g + \int_a^c h = \int_a^b f + \int_b^c f.
\]
\[\square\]

Corollary 28. Let $f, g : [a, b] \to \mathbb{R}$, and suppose $f = g$ except at finitely many points. If $g$ is integrable, then so is $f$, and $\int_a^b f = \int_a^b g$.

\[^2\text{although it would be difficult to compute the integral!}\]
Corollary 29. If \( f : [a, b] \to \mathbb{R} \) is bounded and has only finitely many discontinuities, then \( f \) is integrable.

Theorem 30 (Fundamental Theorem of Calculus). Let \( f \) be integrable on \([a, b]\).

1. Define \( F : [a, b] \to \mathbb{R} \) by \( F(x) = \int_a^x f \). For all \( t \in [a, b] \), if \( f \) is continuous at \( t \) then \( F'(t) = f(t) \).

2. If \( G \) is continuous on \([a, b]\) and \( G' = f \) on \((a, b)\), then \( \int_a^b f = G(b) - G(a) \).

Proof. (1) Given \( \varepsilon > 0 \), choose \( \delta > 0 \) such that for all \( x \in [a, b] \), if \( |x - t| < \delta \) then \( |f(x) - f(t)| < \varepsilon/2 \), so that if also \( x \neq t \) then we have

\[
\left| \frac{F(x) - F(t)}{x - t} - f(t) \right| = \left| \frac{\int_a^x f - \int_a^t f}{x - t} - f(t) \right|
\]

\[
= \left| \int_t^x f(s) \frac{ds}{x - t} - \int_t^t f(t) \frac{ds}{x - t} \right|
\]

\[
= \left| \int_t^x (f(s) - f(t)) \frac{ds}{x - t} \right|
\]

\[
\leq \int_t^x |f(s) - f(t)| \frac{ds}{|x - t|}
\]

\[
\leq \frac{\varepsilon}{2} < \varepsilon.
\]

(2) Let \( g \) be a step function with \( g \leq f \). Without loss of generality (because it will not affect \( \int f \)) we have

\[
a = a_1 < b_1 = a_2 < b_2 = \cdots = a_n < b_n = b
\]

and

\[
g = \sum_{i=1}^n c_i \chi_{A_i},
\]

where \( A_i \) is an interval with endpoints \( a_n \) and \( b_n \). By the Mean Value Theorem, for each \( i \) there exists \( t_i \in (a_i, b_i) \) such that \( G(b_i) - G(a_i) = f(t_i) \ell(A_i) \), so that

\[
G(b) - G(a) = \sum_{i=1}^n (G(b_i) - G(a_i)) = \sum_{i=1}^n f(t_i) \ell(A_i) \geq \sum_{i=1}^n c_i \ell(A_i) = \int g.
\]

Taking the sup over \( g \), we get

\[
G(b) - G(a) \geq \int_a^b f.
\]

Since a similar argument shows \( G(b) - G(a) \leq \int_a^b f \), we are done. \( \square \)
Corollary 31 (Mean Value Theorem for Integrals). If \( f \) is bounded on \([a, b]\) and continuous on \((a, b)\), then there exists \( c \in (a, b) \) such that

\[
\int_a^b f = f(c)(b - a).
\]

Proof. Note that since \( f \) is bounded and has only finitely many discontinuities, it is integrable. Define \( F : [a, b] \to \mathbb{R} \) by \( F(x) = \int_a^x f \). Then \( F \) is continuous on \([a, b]\), and, by the Fundamental Theorem of Calculus, \( F \) is differentiable on \((a, b)\), with \( F' = f \) there. By the Mean Value Theorem for derivatives, there exists \( c \in (a, b) \) such that

\[
\int_a^b f = F(b) - F(a) = F'(c)(b-a) = f(c)(b-a).
\]

Theorem 32 (Integration by Parts). If \( f' \) and \( g' \) are integrable on \([a, b]\), then

\[
\int_a^b f'g = f(b)g(b) - f(a)g(a) - \int_a^b fg'.
\]

Proof. Since \( f \) and \( g \) are differentiable, they are continuous, hence integrable. Thus \( fg' \) and \( f'g \) are integrable. By the Fundamental Theorem of Calculus,

\[
f(b)g(b) - f(a)g(a) = \int_a^b (fg)' = \int_a^b fg' + \int_a^b f'g. \tag*{□}
\]

Theorem 33 (Change of Variables Theorem). If \( \phi' \) is integrable on \([a, b]\) and \( f \) is continuous on \( \phi([a, b]) \) then

\[
\int_a^b f(\phi(x))\phi'(x) \, dx = \int_{\phi(a)}^{\phi(b)} f(u) \, du.
\]

Proof. Since \( \phi \) is differentiable on \([a, b]\), it’s continuous there, so \( \phi([a, b]) \) is a compact interval. Define \( F : \phi([a, b]) \to \mathbb{R} \) by \( F(x) = \int_{\phi(a)}^{x} f \). Then by the Fundamental Theorem of Calculus we have \( F' = f \) on \( \phi([a, b]) \) and \( (F \circ \phi)' = (f \circ \phi)\phi' \) on \([a, b]\) (where we used the Chain Rule for the latter). Since \( \phi \) and \( f \) are continuous, the composition \( f \circ \phi \) is continuous, hence integrable. Since \( \phi' \) is also integrable, the product \((f \circ \phi)\phi'\) is integrable. We apply the (other part of the) Fundamental Theorem of Calculus (twice) in the following computation to finish:

\[
\int_{\phi(a)}^{\phi(b)} f = F \circ \phi(b) - F \circ \phi(a) = \int_{a}^{b} (F \circ \phi)' = \int_{a}^{b} (f \circ \phi)\phi'. \tag*{□}
\]

Improper integrals. Now for a rather curious variation on integrals: “improper” integrals. You’ll remember from your calculus course that these come in two kinds: either the integrand or the interval is unbounded. It turns out that we can deal with both kinds in a unified way.

First, let’s observe one more thing about integrals: let \(-\infty < a < b < \infty\), and suppose \( f : [a, b] \to \mathbb{R} \) is bounded, and integrable on \([a, t]\) for every \( t \in (a, b) \). If \( f(b) \) is defined, we’ve seen before that \( f \) is integrable on \([a, b]\) and \( \int_a^b f = \lim_{t \uparrow b} \int_a^t f \). What if \( b \) is not in the domain of \( f \)? Hopefully you’ve decided right away that there is no need for a separate definition for such a situation: we should just imagine that we’ve extended \( f \) to \([a, b]\) by defining \( f(b) \) any way we want (it doesn’t matter what value we give it), and then as before...
we have \( \int_a^b f = \lim_{t \uparrow b} \int_a^t f \), which only involves the original function \( f \) defined on the right-half-open interval \([a, b)\) — and of course similarly at the left endpoint. Conclusion: we do not want to call such integrals improper, even though the original function is not defined at one endpoint.

**Definition 34.**

(1) Let \( f \) be integrable on \([a, t]\) for every \( t \in (a, b) \). If either \( f \) is unbounded on \([a, b)\) or \( b = \infty \) we define
\[
\int_a^b f = \lim_{t \uparrow b} \int_a^t f.
\]

(2) Let \( f \) be integrable on \([t, b]\) for every \( t \in (a, b) \). If either \( f \) is unbounded on \((a, b]\) or \( a = -\infty \) we define
\[
\int_a^b f = \lim_{t \downarrow a} \int_t^b f.
\]

(3) In either of the two above cases we call \( \int_a^b f \) an improper integral, and if the limit exists we say \( \int_a^b f \) converges, or \( f \) is improperly integrable on \((a, b)\); otherwise we say \( \int_a^b f \) diverges.

(4) More generally, if \( a = x_0 < x_1 < \cdots < x_n = b \) and if for each \( i = 1, \ldots, n \) either \( f \) is integrable on \([x_{i-1}, x_i]\) or the integral \( \int_{x_{i-1}}^{x_i} f \) is improper, with at least one of the integrals \( \int_{x_{i-1}}^{x_i} f \) for \( i = 1, \ldots, n \) being improper, then we call \( \int_a^b f \) an improper integral, and moreover if for every \( i = 1, \ldots, n \) either \( f \) is integrable on \([x_{i-1}, x_i]\) or improperly integrable on \((x_{i-1}, x_i)\), then we define
\[
\int_a^b f = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f,
\]
and we say \( f \) is improperly integrable on \((a, b)\), or \( \int_a^b f \) converges; otherwise we say \( \int_a^b f \) diverges.

**Observation 35.** In case (1), if \( c \in (a, b) \) then \( \int_a^b f \) exists if and only if \( \int_c^b f \) does, in which case \( \int_a^b f = \int_a^c f + \int_c^b f \). Similarly in case (2).

**Examples.**

(1) Both \( \int_1^\infty x \, dx \) and \( \int_1^\infty 1/x^2 \, dx \) are improper integrals; the first diverges and the second converges.

(2) \( \int_0^\infty 1/x \, dx \) is improper because both \( \int_1^1 1/x \, dx \) and \( \int_1^\infty 1/x \, dx \) are (albeit for different reasons).

(3) \( \int_{-\infty}^\infty 1/x \, dx \) is improper because \( \int_{-\infty}^{-1} 1/x \, dx \), \( \int_{-1}^0 1/x \, dx \), and \( \int_0^\infty 1/x \, dx \) are.

**Proposition 36** (Arithmetic of Improper Integrals). If each of \( f \) and \( g \) is either Riemann or improperly integrable, then

\[3]I have to comment that I find this definition ultimately unsatisfying (although I like it better than any other I’ve read), however I have come to the sad conclusion that with the accepted terminology there’s no way to define improper integrals in a logically reassuring way. Roughly speaking, improper integrals are certain limits of integrals.
\begin{align*}
(1) \quad & \int_a^b (f + g) = \int_a^b f + \int_a^b g; \\
(2) \quad & \int_a^b cf = c \int_a^b f \text{ if } c \in \mathbb{R}.
\end{align*}

More precisely, we’re assuming that both \( f \) and \( g \) are improperly integrable on \((a, b)\), or \(-\infty < a < b < \infty\) and one of \( f \) or \( g \) is integrable on \([a, b]\) while the other is either improperly integrable on \((a, b)\) or integrable on \([a, b]\) — note that this actually allows both to be integrable, although we do not get a new proof of the arithmetic properties of integrals.

Proof. If necessary, partition the interval \([a, b]\), so that without loss of generality\(^4\) both \( f \) and \( g \) are integrable on \([a, t]\) for all \( t \in (a, b)\). For (1), we have

\[
\int_a^b (f + g) = \lim_{t \uparrow b} \int_a^t (f + g) = \lim_{t \uparrow b} \left( \int_a^b f + \int_a^b g \right) \\
= \lim_{t \uparrow b} \int_a^b f + \lim_{t \uparrow b} \int_a^b g \\
= \int_a^b f + \int_a^b g.
\]

For (2), we have

\[
\int_a^b cf = \lim_{t \uparrow b} \int_a^t cf = \lim_{t \uparrow b} c \int_a^t f = c \lim_{t \uparrow b} \int_a^t f = c \int_a^b f.
\]

\[\Box\]

**Theorem 37** (Comparison Theorem for Improper Integrals). If \( \int_a^b f \) is improper, \( g \) is improperly integrable on \((a, b)\), and \( |f| \leq g \), then \( f \) is improperly integrable and

\[
\left| \int_a^b f \right| \leq \int_a^b g.
\]

Proof. If necessary, partition the interval \([a, b]\), so that without loss of generality \( f \) and \( g \) are integrable on \([a, t]\) for all \( t \in (a, b)\). First assume that \( f \geq 0 \). Then the function \( t \mapsto \int_a^t f \) is increasing. Since \( f \leq g \) we have

\[
\int_a^t f \leq \int_a^t g \leq \int_a^b g \quad \text{for all } t \in (a, b).
\]

Thus the increasing function \( t \mapsto \int_a^t f \) is bounded above, so the left hand limit \( \lim_{t \uparrow b} \int_a^t f \) exists, hence \( f \) is improperly integrable.

Now let \( f \) be arbitrary. Since

\[
0 \leq f + g \leq 2g
\]

and \( 2g \) is improperly integrable, by the above we see that \( f + g \) is improperly integrable, hence so is \( f = (f + g) - g \).

For the other part, we have

\[
\left| \int_a^t f \right| \leq \int_a^t |f| \leq \int_a^t g \quad \text{for all } t \in (a, b).
\]

\(^4\) because the other “basic” type is handled similarly, and the general result follows from adding over the subintervals of the partition.
so letting \( t \uparrow b \) we get

\[
\left| \int_a^b f \right| \leq \int_a^b g.
\]

\( \square \)

**Example.** \( \int_1^\infty \frac{1}{x^p} \, dx \) exists if and only if \( p > 1 \). To verify this, first let \( p > 1 \). Then

\[
\lim_{t \to \infty} \frac{x^{1-p}}{1-p} \bigg|_1^t = \frac{1}{p-1},
\]

since \( \lim_{t \to \infty} t^a = 0 \) when \( a < 0 \). For \( p = 1 \),

\[
\lim_{t \to \infty} \log x \bigg|_1^t = \infty,
\]

so \( \int_1^\infty \frac{1}{x} \, dx \) does not exist. Finally, for \( p < 1 \) we have \( \frac{1}{x^p} \geq \frac{1}{x} \) for all \( x \geq 1 \), so by the Comparison Theorem \( \int_1^\infty \frac{1}{x^p} \, dx \) fails to exist.