Exercise 1. Prove that the continuous functions $x^n$ converge pointwise on $[0, 1]$ (as $n \to \infty$) to a discontinuous limit.

Exercise 2. Find an example of a subset $A \subset \mathbb{R}$ and a sequence $(f_n)$ of functions from $A$ to $\mathbb{R}$ such that:

• every $f_n$ is continuous,
• $(f_n)$ converges pointwise but not uniformly, and
• $\lim f_n$ is continuous.

Exercise 3. For each $n \in \mathbb{N}$ define $f_n : [0, 1] \to \mathbb{R}$ by

\[ f_n(x) = \begin{cases} 
  n & \text{if } 0 < x < \frac{1}{n} \\
  0 & \text{if } x = 0 \text{ or } \frac{1}{n} \leq x \leq 1.
\end{cases} \]

Show that $(f_n)$ converges pointwise to an integrable function, but

\[ \int_0^1 \lim f_n \neq \lim \int_0^1 f_n. \]

Exercise 4. Find an example of a sequence $(f_n)$ of continuous functions from $[0, 1]$ to $\mathbb{R}$ which converges pointwise to 0 but for which $\int_0^1 f_n \neq 0$.

Exercise 5. Let $g : \mathbb{R} \to \mathbb{R}$. Suppose:

• $g$ is differentiable,
• $g(0) = 0$,
• $\lim_{x \to \pm \infty} g(x) = 0$, and
• $g'(0) \neq 0$.

For each $n \in \mathbb{N}$ define $f_n : \mathbb{R} \to \mathbb{R}$ by

\[ f_n(x) = \frac{g(nx)}{n}. \]

Prove that $f_n \to 0$ uniformly, but $f_n'(0) \neq 0$. 

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Exercise 6. Let $A \subset \mathbb{R}$, let $(f_n)$ be a uniformly convergent sequence of functions from $A \rightarrow \mathbb{R}$, let $(x_n)$ be a sequence in $A$, and let $u \in \mathbb{R}$. Put $f = \lim f_n$. Prove that if $f(x_n) \rightarrow u$, then we also have $f_n(x_n) \rightarrow u$.

Exercise 7. Let $A \subset \mathbb{R}$, and let $(f_n)$ and $(g_n)$ be uniformly convergent sequences of functions from $A$ to $\mathbb{R}$. Prove that $(f_n + g_n)$ converges uniformly.

Exercise 8. Let $A \subset \mathbb{R}$, and let $(f_n)$ and $(g_n)$ be uniformly convergent sequences of functions from $A$ to $\mathbb{R}$. Suppose that there exists $M > 0$ such that for all $n \in \mathbb{N}$ and $x \in A$ we have $|f_n(x)| \leq M$ and $|g_n(x)| \leq M$.

Prove that $(f_n g_n)$ converges uniformly.

Exercise 9. Show that the functions $x + 1/n$ converge uniformly on $\mathbb{R}$ (as $n \rightarrow \infty$), but $(x + 1/n)^2$ does not converge uniformly.

Exercise 10. Prove that the series
\[ \sum_{n=1}^{\infty} \frac{1}{x + n^2} \]
is differentiable on $[0, \infty)$, and find a formula for the derivative.