Exercise 1. Let \( \sigma : \mathbb{R} \to \mathbb{R} \) be a solution of the differential equation
\[
\sigma'' = -\sigma
\]
such that \( \sigma(0) = 0 \) and \( \sigma'(0) = 1 \). Put \( \kappa = \sigma' \). Prove that for all \( x \in \mathbb{R} \) we have
\[
(\sigma(x))^2 + (\kappa(x))^2 = 1.
\]

Exercise 2. With the notation from the preceding exercise, let
\[
sin = \sigma \quad \text{and} \quad \cos = \kappa,
\]
and use the customary notation for these two trig functions. Let \( f \) be any solution of the differential equation (1) above. Prove that
\[
f = f'(0) \sin + f(0) \cos.
\]
Hint: Show that both \( f \sin + f' \cos \) and \( f \cos - f' \sin \) are constant. Consider \( \sin \) times the 1st plus \( \cos \) times the 2nd.

Exercise 3. Prove the following identities:

(a) \( \sin(-x) = -\sin x \);
(b) \( \cos(-x) = \cos x \);
(c) \( \sin(x + y) = \sin x \cos y + \cos x \sin y \);
(d) \( \cos(x + y) = \cos x \cos y - \sin x \sin y \);
(e) \( \sin(2x) = 2 \sin x \cos x \);
(f) \( \cos(2x) = \cos^2 x - \sin^2 x \);

Exercise 4. Prove that there exists \( x > 0 \) such that \( \cos x = 0 \).

Hint: Suppose not, and deduce that:

(a) for all \( x > 0 \) we have \( 0 < \cos x < 1 \);
(b) putting \( a = \cos 1 \), for all \( n \in \mathbb{N} \) we have \( \cos(2^n) \leq a^{2^n} \);
(c) there exists \( t > 0 \) such that \( \cos t < 1/2 \);
(d) for the above \( t \) we have \( \cos(2t) < 0 \).

Exercise 5. Prove that the set \( \{ x > 0 : \cos x = 0 \} \) has a minimum.

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Exercise 6. Let \( \pi \) denote 2 times the minimum from the preceding exercise. Prove the following identities:

(a) \( \sin \left( x + \frac{\pi}{2} \right) = \cos x; \)
(b) \( \cos \left( x + \frac{\pi}{2} \right) = -\sin x; \)
(c) \( \sin(x + \pi) = -\sin x; \)
(d) \( \cos(x + \pi) = -\cos x; \)
(e) \( \sin(x + 2\pi) = \sin x; \)
(f) \( \cos(x + 2\pi) = \cos x. \)

Exercise 7. Prove that the function \( \cos \) maps the interval \([0, \pi]\) 1-1 onto the interval \([-1, 1]\).

Exercise 8. Let

\[ A = \left\{ \frac{(2n + 1)\pi}{2} : n \in \mathbb{Z} \right\}, \]

and define \( \tan : \mathbb{R} \setminus A \to \mathbb{R} \) by

\[ \tan = \frac{\sin}{\cos}. \]

Prove:

(a) \( \frac{d}{dx} \tan x = \frac{1}{\cos^2 x}. \)

(b) For all \( x \in A \) we have \( \tan(x + \pi) = \tan x. \)

(c) \( \tan \) is strictly increasing on the interval \((-\pi/2, \pi/2).\)

(d) \( \tan \) has range \( \mathbb{R}. \)