Exercise 1. Let $A \subset \mathbb{R}$, $f: A \to \mathbb{R}$, and $t \in A$. Suppose $t$ is a cluster point of $A$. Prove that $f$ is continuous at $t$ if and only if $\lim_{x \to t} f(x) = f(t)$.

Exercise 2. Let $A \subset \mathbb{R}$, $f: A \to \mathbb{R}$, and $t \in A$. Prove that if $t$ is not a cluster point of $A$, then $f$ is continuous at $t$.

Exercise 3. Let $A \subset \mathbb{R}$, $f: A \to A$, and $s, t \in A$. Define a sequence $(x_n)$ inductively by

$$x_n = \begin{cases} s & \text{if } n = 1 \\ f(x_{n-1}) & \text{if } n > 1. \end{cases}$$

Assume that $f$ is continuous and that $x_n \to t$. Prove that $f(t) = t$.

Exercise 4. Let $A \subset \mathbb{R}$, and let $f$ and $g$ be uniformly continuous functions from $A$ to $\mathbb{R}$. Prove that $f + g$ is uniformly continuous.

Exercise 5. Let $A \subset \mathbb{R}$, and let $f$ and $g$ be uniformly continuous functions from $A$ to $\mathbb{R}$. Suppose that both $f$ and $g$ are bounded. Prove that $fg$ is uniformly continuous.

Exercise 6. Prove that $x$ is uniformly continuous on $\mathbb{R}$, but $x^2$ is not.

Exercise 7. Let $f: \mathbb{R} \to (0, \infty)$ be continuous, and assume that

$$\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = 0.$$ 

Prove that $f$ has a maximum.

Hint: show that there exists $a > 0$ such that for all $x \in \mathbb{R}$, if $|x| > a$ then $f(x) < f(0)$.

Exercise 8. Let $f: [0, 1] \to [0, 1]$ be continuous. Prove that there exists $c \in [0, 1]$ such that $f(c) = c$.

Exercise 9. (Roots) Use the Intermediate Value Theorem to prove that for every $n \in \mathbb{N}$, every nonnegative real number has a unique $n$th root.

Exercise 10. Prove that for every $n \in \mathbb{N}$, the function $x^{1/n}$ on $[0, \infty)$ is continuous.