Name:

- No books, notes or calculators of any kind are permitted on the test.
- Do not use your own scratch paper.
- Write each solution in the space provided, not on scratch paper.
- If you need more room, write on the back of the page. If you still need more room, ask for scratch paper.
- Be sure to give reasons for your answers (except where explicitly told otherwise)!
- Your solutions must be complete and organized, otherwise points may be deducted.
- There are 5 problems, and all problems have equal credit.
1. Complete the following definitions:

(a) Let $X$ be a metric space, let $A \subset X$, and let $t \in X$.
We say $t$ is a cluster point of $A$ if

(b) Let $X$ and $Y$ be metric spaces, let $f : X \to Y$, and let $t \in X$.
We say $f$ is continuous at $t$ if

(c) Let $(x_n)$ be a sequence in a metric space $X$.
We say $(x_n)$ is Cauchy if
2. State the following theorems:

(a) Baire Category Theorem

(b) Bolzano-Weierstrass Theorem (for $\mathbb{R}^n$)

(c) Lindelöf’s Theorem
3. Let $X$ be a metric space. For $A \subset X$, define
\[ \text{diam}(A) = \sup\{d(x, y) \mid x, y \in A\}. \]

Let $A_1 \supset A_2 \supset \cdots$ be a decreasing sequence of closed nonempty subsets of $X$. Prove that if $X$ is complete and $\text{diam}(A_n) \to 0$ then $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$. 

4. In this problem, only the answer will be graded; you do not need to show any work!

(a) Find \( \lim_{x \to 0} x \sin \left( \frac{1}{x} \right) \).

Solution: 0

(b) Does there exist a sequence whose range is \( \mathbb{R} \)?

Solution: no

(c) If \( X \) and \( Y \) are metric spaces, \( f: X \to Y \) is continuous, and \( B \subset Y \) is closed, must \( f^{-1}(B) \) be closed in \( X \)?

Solution: yes

(d) Find an example of a set \( A \) which is open in the metric space \([0, 1]\) (regarded as a subspace of \( \mathbb{R} \)) but not open in \( \mathbb{R} \).

Solution: \( A = [0, 1/2) \)

(e) Find \( \lim \sup \frac{(-1)^n n}{n + 1} \).

Solution: 1

(f) Find \( \lim_{x \to \infty} \frac{x^3 - x}{x + 1} \).

Solution: \( \infty \)
5. Let $X$ and $Y$ be metric spaces, let $f: X \to Y$ be continuous, and let $A \subset X$. Prove that $\overline{f(A)}$. 

Let $x \in A$. Then there exists a sequence $(x_n)$ in $A$ converging to $x$. By continuity, $f(x_n) \to f(x)$. Since $f(x_n) \in f(A)$ for all $n \in \mathbb{N}$, we have $f(x) \in \overline{f(A)}$. 

$$f\left(\overline{A}\right) \subset \overline{f(A)}.$$