MAT 472 Final Exam   Due: 5PM December 14, 1999
Instructor: John Quigg

- **No** notes, books, or calculators.
- You must show **all** your work to receive credit.
- Your solutions must be **complete** and **organized**, and your **final answer** must be **clearly** indicated.
- Write your solutions on **blank** pages, preferably white 8.5-by-11 inch, on **one side only**, leaving a 1-inch margin on all sides, and put your **name** at the **top right corner** of every page.
- This exam has **2 pages**.
- All problems have equal credit.
- Do **not** turn in either this page or the exam question page.
- **Staple** the pages of your exam; do not fold the corner of the exam over.
1. Let 

\[ f(x) = \frac{\sin x}{x}. \]

Prove that \( \int_{2\pi}^{\infty} f \) exists, but \( \int_{2\pi}^{\infty} |f| \) does not.

2. For each \( n \in \mathbb{N} \) put 

\[ A_n := \{ f \in C[0,1] : |f(x) - f(y)| \leq n|x - y| \text{ for all } x, y \in [0,1] \}. \]

(a) Prove that for each \( n \in \mathbb{N} \) the set \( A_n \) is closed in the metric space \( C[0,1] \).

(b) Prove that for each \( n \in \mathbb{N} \) and \( r \geq 0 \) the subset \( \{ f \in A_n : |f(0)| \leq r \} \) of \( C[0,1] \) is compact.

3. (a) With \( A_n \) as in the preceding problem, prove that the complement of \( A_n \) in \( C[0,1] \) is dense in \( C[0,1] \).

(b) A function \( f \) on \([0,1]\) is called Lipschitz if there exists \( M \in \mathbb{R} \) such that \( |f(x) - f(y)| \leq M|x - y| \) for all \( x, y \in [0,1] \). What do the results of part (a) of both this problem and the preceding problem, together with the Baire Category Theorem, allow you to conclude about the set of Lipschitz functions? Be sure to justify all your assertions.

4. Let \( X \) and \( Y \) be metric spaces, and let \( f: X \to Y \) be uniformly continuous. Prove that if \( (x_n) \) is a Cauchy sequence in \( X \) then the sequence \( (f(x_n)) \) is also Cauchy.

5. Let \( f: \mathbb{R} \to \mathbb{R} \) and \( a \in \mathbb{R} \).

(a) Suppose that \( f''(a) \) exists (in particular, \( f \) is differentiable on some open interval containing \( a \)). Prove that there exists a unique polynomial \( h \) of degree at most two such that

\[ \lim_{x \to a} \frac{f(x) - h(x)}{(x - a)^2} = 0. \]

(b) Suppose instead that \( f \) is continuous at \( a \), and that there exists a polynomial \( g \) such that

\[ \lim_{x \to a} \frac{f(x) - g(x)}{(x - a)^2} = 0. \]

Prove that \( f \) is differentiable at \( a \).

6. Prove that

\[ \sum_{n=0}^{\infty} \int_{0}^{1} \frac{x^n}{n!} e^{-x} \, dx = 1. \]

Justify your steps!