MAT 444 Midterm  
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(1) (10 points)
(a) What are the simple groups (up to isomorphism) of order less than 10?
(b) What nonabelian group has the smallest number of elements?
(c) What nonabelian simple group has the smallest number of elements?
(d) Up to isomorphism, how many infinite cyclic groups are there?
(e) Up to isomorphism, how many groups of order 8 are there?

(2) (4 points) Let $G$ be a cyclic group of order 4.
   (a) If $G$ acts on itself by left multiplication, how many orbits are there?
   (b) If $G$ acts on itself by conjugation, how many orbits are there?

(3) (9 points) Let $G$ be a group of order $p^n$, where $p$ is prime and $n \in \mathbb{N}$.
   (a) Must $G$ have nontrivial center?
   (b) Must $G$ be abelian if $n = 2$?
   (c) Must $G$ be abelian if $n = 3$?

(4) (7 points)
   (a) How many elements of order 12 does $S_3 \times Z_4$ have?
   (b) How many elements of order 6 does $S_3 \times Z_4$ have?
   (c) How many elements of order 4 does $S_3 \times Z_4$ have?
   (d) How many elements of order 3 does $S_3 \times Z_4$ have?
   (e) How many elements of order 2 does $S_3 \times Z_4$ have?
   (f) How many elements of order 1 does $S_3 \times Z_4$ have?
   (g) How can you use the numbers found in the above parts to determine how many homomorphisms there are from $Z_{12}$ to $S_3 \times Z_4$?

(5) (14 points) Consider the subgroup $N = \{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$ of $S_4$. Also consider the elements $a = (1,2)$ and $b = (2,3,4)$ of $S_4$.
   (a) What are the orders of the elements of $N$?
   (b) To what famous group is $N$ isomorphic?
   (c) What is the conjugacy class of the element $(1,2)(3,4)$ of $S_4$?
   (d) Why is $N$ normal in $S_4$?
   (e) Compute the product $aNbN$ in the quotient group $S_4/N$.
   (f) Compute the product $bNaN$ in the quotient group $S_4/N$.
   (g) To what famous group is $S_4/N$ isomorphic?

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(6) (14 points) In this problem you will do some analysis on the group $D_{14}$. You must use Sylow theory to answer parts (b)–(g) - no actual computing with elements allowed!

(a) What is the order of $D_{14}$?

(b) Prove that $D_{14}$ has only 1 Sylow 7-subgroup.

(c) Why must the Sylow 7-subgroup be normal in $D_{14}$?

(d) Let $N$ be the Sylow 7-subgroup, and let $H$ be a Sylow 2-subgroup. What is the order of $H \cap N$?

(e) With $H, N$ as above, what is the order of $HN$?

(f) Prove that $D_{14}$ has more than 1 Sylow 2-subgroup. You may use the fact that if $H, N \triangleleft D_{14}$ and $H \cap N = \{e\}$, then the subgroup of $D_{14}$ generated by $H$ and $N$ is isomorphic to $H \times N$.

(g) How many subgroups of order 4 does $D_{14}$ have?