UNIQUENESS OF JORDAN FORM

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1. Preliminaries

Notation and Terminology. (i) $V$ is a finite-dimensional vector space and $T \in \mathcal{L}(V)$.
(ii) $\mathcal{S}(T)$ is the spectrum of $T$.

Definition. A basis $\beta$ of $V$ is called a Jordan basis for $T$, and $[T]_{\beta}$ is called a Jordan form of $T$, if there exist:
(i) a finite set $\Lambda \subset \mathbb{C}$,
(ii) a partition $\{\beta_\lambda\}_{\lambda \in \Lambda}$ of $\beta$, and
(iii) for each $\lambda \in \Lambda$ a partition $\Gamma_\lambda$ of $\beta_\lambda$,
such that for each $\lambda \in \Lambda$ the subspace $\langle \beta_\lambda \rangle$ is $T$-invariant and

$$[T|\langle \beta_\lambda \rangle]_{\beta_\lambda} = \bigoplus_{\gamma \in \Gamma_\lambda} J_\gamma,$$

where each $J_\gamma$ is of the form

$$ \begin{pmatrix} \lambda & 1 & 0 \\ \lambda & 1 & \ddots \\ 0 & \ddots & \ddots & \ddots \\ & & & \lambda & 1 \end{pmatrix}. $$

Each such $J_\gamma$ is called a Jordan block of $T$ associated to $\lambda$.

Remark. A $1 \times 1$ Jordan block associated to $\lambda$ is just $(\lambda)$.

Recall that there exists $p \in \mathbb{N}$ such that $NT^p = \bigcup_{k=1}^{\infty} NT^k$, and then $V = NT^p \oplus RT^p$ is a $T$-invariant decomposition such that $T$ is nilpotent on $NT^p$ and invertible on $RT^p$. We now show this decomposition is unique:

Lemma. Let $V = W \oplus Z$ be a $T$-invariant decomposition, with $T$ nilpotent on $W$ and invertible on $Z$. Then, with the above notation, $W = NT^p$ and $Z = RT^p$.

Proof. By hypothesis, there exists $k \in \mathbb{N}$ such that $T^k = 0$ on $W$, so

$$W \subset NT^p.$$

Also, by hypothesis $T(Z) = Z$, so $T^p(Z) = Z$, hence

$$Z \subset RT^p.$$

Thus

$$\dim W \leq \text{nullity} T^p \quad \text{and} \quad \dim Z \leq \text{rank} T^p.$$

But then

$$\dim V = \text{nullity} T^p + \text{rank} T^p \geq \dim W + \dim Z = \dim V,$$

so we must have equality throughout. Therefore

$$\dim W = \text{nullity} T^p \quad \text{and} \quad \dim Z = \text{rank} T^p.$$
so

\[ W = NT^p \quad \text{and} \quad Z = RT^p. \]

\[ \square \]

**Uniqueness Theorem A.** Let \( \beta \) be a Jordan basis for \( T \). Then, with the above notation:

(i) \( \Lambda = S(T) \);
(ii) \( \langle \beta_\lambda \rangle = K_\lambda(T) \) for all \( \lambda \in \Lambda \).

**Proof.** (i) By definition, \( [T]_\beta \) is upper triangular, so \( S(T) \) is the set of scalars on the diagonal of \( [T]_\beta \), which is precisely \( \Lambda \).

(ii) Let \( \lambda \in \Lambda \). Then by construction \( T - \lambda I \) is nilpotent on \( \langle \beta_\lambda \rangle \) and invertible on \( \sum_{\mu \neq \lambda} \langle \beta_\mu \rangle \). By the preceding lemma,

\[ \langle \beta_\lambda \rangle = \bigcup_{k=1}^\infty N(T - \lambda I)^k. \]

but by earlier work we recognize that this latter subspace is \( K_\lambda(T) \). \[ \square \]

For the uniqueness of the Jordan blocks, we simplify things by reducing to the situation where there is only 1 eigenvalue, and moreover we add a multiple of the identity to make this eigenvalue 0:

**Uniqueness Theorem B.** Let \( T \) be nilpotent, and let \( \beta \) be a Jordan basis for \( T \). Then:

(i) The number of Jordan blocks is nullity \( T \);
(ii) For each \( k = 2, 3, \ldots \) the number of Jordan blocks of size at least \( k \) is nullity \( T^k \) – nullity \( T^{k-1} \).

**Proof.** We have a partition \( \Gamma \) of \( \beta \) such that

\[ [T]_\beta = \bigoplus_{\gamma \in \Gamma} J_\gamma, \]

where each \( J_\gamma \) is of the form

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 1 & 0 \\
\vdots & \ddots & \ddots \\
0 & \cdots & 1 & 0
\end{pmatrix}.
\]

This means each \( \gamma \) is a cycle of \( T \), and \( J_\gamma \) is of size \( \# \gamma \) (the number of elements in \( \gamma \)). For each \( k \in \mathbb{N} \) put

\[ \zeta_k = \beta \cap NT^k \]

and

\[ \Gamma_k = \{ \gamma \in \Gamma : \# \gamma \geq k \}. \]

Then

\[ T^k(\beta \setminus \zeta_k) = T^k(\beta) \setminus \{0\} = \beta \cap RT^k \]

is a basis for \( RT^k \), and moreover \( T^k \) maps \( \beta \setminus \zeta_k \) 1-1 into \( \beta \). Thus

\[ \# \zeta_k = \# \beta - \#(\beta \setminus \zeta_k) = \dim V - \text{rank } T^k = \text{nullity } T^k. \]

When \( k = 1 \) this gives (i).

Suppose \( k \geq 2 \). Note that for all \( \gamma \in \Gamma \) we have

\[ \gamma \in \Gamma_k \iff \gamma \cap (\zeta_k \setminus \zeta_{k-1}) \neq \emptyset, \]

and moreover \( \zeta_k \setminus \zeta_{k-1} \) contains precisely 1 element from each \( \gamma \in \Gamma_k \). Thus

\[ \# \Gamma_k = \#(\zeta_k \setminus \zeta_{k-1}) = \# \zeta_k - \# \zeta_{k-1} = \text{nullity } T^k - \text{nullity } T^{k-1}. \]

\[ \square \]
Corollary. Let $J$ be a Jordan form for $T$, where $T$ is no longer assumed nilpotent. Then for each eigenvalue $\lambda$ we have:

(i) The number of Jordan blocks associated to $\lambda$ is $\text{nullity}(T - \lambda I)$;

(ii) For each $k = 2, 3, \ldots$ the number of Jordan blocks associated to $\lambda$ which have size at least $k$ is $\text{nullity}(T - \lambda I)^k - \text{nullity}(T - \lambda I)^{k-1}$.

Remark. In view of the above, if we order the Jordan basis so that for each eigenvalue $\lambda$ the sizes of the associated Jordan blocks decrease down the (block) diagonal, the Jordan form is unique up the ordering of the eigenvalues. It is in this sense that we refer to the Jordan form of $T$.

Definition. Let $A \in M_n(\mathbb{F})$, and assume that the characteristic polynomial of $A$ splits. Then the Jordan form of $L_A$ is also called the Jordan form of $A$.

Corollary. Let $A, B \in M_n(\mathbb{F})$, and assume that the characteristic polynomials of both $A$ and $B$ split. Then $A$ and $B$ are similar if and only if they have the same Jordan form (up to the ordering of the eigenvalues).

Proof. Let $J_A$ and $J_B$ be the Jordan forms of $A$ and $B$, respectively. Then $A$ is similar to $J_A$ and $B$ is similar to $J_B$. Also, by uniqueness of Jordan form, $J_A$ and $J_B$ are similar if and only if they agree up to ordering of the eigenvalues. Thus $A$ and $B$ are similar if and only if $J_A$ and $J_B$ agree up to ordering of the eigenvalues. \qed