21. Series of functions

Exercise 21.1. Prove that $\sum_{n=0}^{\infty} x^n$ converges uniformly on every interval of the form $[-s, s]$ with $0 < s < 1$.

Exercise 21.2. Find an example of a series of functions which converges uniformly on $\mathbb{R}$ but does not satisfy the hypotheses of the Weierstrass $M$-Test.

Exercise 21.3. Prove:

(a) $\sum_{n=1}^{\infty} \frac{1}{(x+n^2)^2}$ converges uniformly on $(0,1)$.

(b) $\sum_{n=1}^{\infty} \frac{1}{x+n^2}$ converges on $(0,1)$ to a differentiable function, and

$$\frac{d}{dx} \sum_{n=1}^{\infty} \frac{1}{x+n^2} = -\sum_{n=1}^{\infty} \frac{1}{(x+n^2)^2}$$

for all $x \in (0,1)$.

Hint: show that the series converges for $x = 1/2$.

Exercise 21.4. Define $f : [0,1] \to \mathbb{R}$ as follows: first of all, $f(x) = 1$ if $x = 0$ and $f(x) = 0$ if $x \not\in \mathbb{Q}$. For every nonzero rational $x \in [0,1]$, there exist unique $k, n \in \mathbb{N}$ such that $x = k/n$ in lowest terms, that is, $k$ and $n$ have no common prime divisors; in this case define $f(x) = 1/n$.

Prove:

(a) The discontinuities of $f$ are precisely the rational numbers in $[0,1]$.

(b) $f$ is integrable.

Hint: define

$$f_0(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if not,} \end{cases}$$

and for each $n \in \mathbb{N}$ define

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{k}{n} \text{ in lowest terms} \\ 0 & \text{if not} \end{cases}$$

Then $f = \sum_{n=0}^{\infty} f_n$, and for all $x \in [0,1]$ we have $f_n(x) \neq 0$ for at most 1 value of $n$. 

1