Exercise 19.1. (Inserting parentheses) Let $\sum_{n=1}^{\infty} a_n$ converge, and let $(n_1, n_2, n_3, \ldots)$ be a strictly increasing sequence of positive integers. Put $n_0 = 0$. For each $j \in \mathbb{N}$ define

$$b_j = a_{n_{j-1}+1} + \cdots + a_{n_j}.$$ 

Prove that $\sum_{j=1}^{\infty} b_j$ converges and

$$\sum_{j=1}^{\infty} b_j = \sum_{n=1}^{\infty} a_n.$$

Exercise 19.2.  

(a) Prove that the series

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} - \int_{n}^{n+1} \frac{1}{x} \, dx \right)$$

converges. Hint: show that

$$0 \leq \frac{1}{n} - \int_{n}^{n+1} \frac{1}{x} \, dx \leq \frac{1}{n} - \frac{1}{n+1}$$

for all $n \in \mathbb{N}$.

(b) Use the result of part (a) to show that Euler’s constant

$$\lim_{k \to \infty} \left( \sum_{n=1}^{k} \frac{1}{n} - \log(k+1) \right)$$

exists. Hint: what are the partial sums of the series from part (a)?

Exercise 19.3. Show that the series

$$\sum_{n=2}^{\infty} \frac{3n^3 - 5n + 1}{7n^4 - 6n^2 + 3n - 8}$$

diverges.

Exercise 19.4. In each part, determine whether the series converges:

(a) 

$$\sum_{n=1}^{\infty} \frac{2 + \sin n}{n^2}$$

(b) 

$$\sum_{n=1}^{\infty} \frac{2 + \cos n}{\sqrt{n}}$$

Exercise 19.5. Let $f : [a, \infty) \to \mathbb{R}$ be integrable on $[a, t]$ for all $t > a$, and suppose $(a_n)_{n=0}^{\infty}$ is a strictly increasing sequence diverging to $\infty$ such that $a_0 = a$.

(a) Prove that if $\int_{a}^{\infty} f$ converges, then so does the series $\sum_{n=1}^{\infty} \int_{a_{n-1}}^{a_n} f$. 

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(b) Assuming \( f \) is nonnegative, prove the converse of part (a).

**Exercise 19.6.** Give an example of a continuous\(^1\) nonnegative function \( f \) on \([1, \infty)\) such that \( \int_1^\infty f \) converges but \( f(x) \not\to 0 \) as \( x \to \infty \).

**Exercise 19.7.** Consider the series \( \sum_{n=1}^\infty a_n \), where
\[
a_n = \begin{cases} 3^{-n} & \text{if } n \text{ is odd} \\ 2^{-n} & \text{if } n \text{ is even}. \end{cases}
\]

(a) Show that the Ratio Test gives no information.
(b) Prove that the Root Test shows that the series converges.

**Exercise 19.8.** Find all nonzero real numbers \( p \) for which the series \( \sum_{n=1}^\infty p^n n^p \) converges absolutely, converges conditionally, or diverges.

**Exercise 19.9.** Let \( a_n > 0 \) for all \( n \in \mathbb{N} \). Prove that if \( \sum a_n \) diverges, then so does \( \sum \frac{a_n}{1 + a_n} \).

**Exercise 19.10.** Does the series
\[
\sum_{n=1}^\infty 2^n e^{-n}
\]
converge?

**Exercise 19.11.** Does the series
\[
\sum_{n=1}^\infty n^n e^{-n}
\]
converge?

**Exercise 19.12.** Does the series
\[
\sum_{n=1}^\infty e^{-\log n}
\]
converge?

**Exercise 19.13.** Does the series
\[
\sum_{n=1}^\infty (\log n)e^{-\sqrt{n}}
\]
converge? (This one’s tricky.)

**Exercise 19.14.** Does the series
\[
\sum_{n=1}^\infty n! e^{-n}
\]
converge?

\(^1\)but not uniformly so, by a previous exercise!
Exercise 19.15. Does the series
\[ \sum_{n=1}^{\infty} n!e^{-n^2} \]
converge?

Exercise 19.16. Does the series
\[ \sum_{n=1}^{\infty} \frac{2^n n!}{n^n} \]
converge?

Exercise 19.17. Does the following series converge?
\[ \sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n} \]

Exercise 19.18. Let \((a_n)\) be a decreasing sequence of positive numbers. Prove that if \(\sum a_n\) converges, then \(na_n \to 0\). Hint: consider partial sums of a sufficiently small tail \(\sum_{k+1}^{\infty} a_n\). (This one’s tricky.)