15. Differentiation

**Exercise 15.1.** Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable at only one point.

**Exercise 15.2.** Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Prove:

(a) $f$ is differentiable;
(b) $f'$ is discontinuous at 0.

**Exercise 15.3.** Let $f, g : (a, b) \to \mathbb{R}$ and $a < c < b$. Suppose $f \leq g$ on $(a, b)$, $f$ and $g$ are both differentiable at $c$, and $f(c) = g(c)$. Use the Critical Point Lemma to prove that $f'(c) = g'(c)$.

**Exercise 15.4.** Use the Mean Value Theorem to prove that $\sqrt{1 + x} < 1 + \frac{x}{2}$ for all $x > 0$.

**Exercise 15.5.** Let $f : (a, b) \to \mathbb{R}$ be differentiable. Prove that if $f'$ is bounded, then $f$ is uniformly continuous. Hint: Mean Value Theorem.

**Exercise 15.6.** Let $f : (a, b) \to \mathbb{R}$ be differentiable. Prove that if $f$ is Lipschitz, then $f'$ is bounded. Hint: Mean Value Theorem.

**Exercise 15.7.** Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^3 + 3x^2 - 36x + 5 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

(a) Prove that $f$ is 1-1 on the interval $[-1, 1]$.
(b) Find $(f^{-1})'(5)$.

**Exercise 15.8.** Let $f : (a, b) \to \mathbb{R}$ and $t \in (a, b)$. Suppose $f$ is continuous at $t$ and differentiable on $(a, b) \setminus \{t\}$, and $\lim_{x \to t} f'(x)$ exists. Prove:

(a) $f$ is differentiable at $t$.
(b) $f'$ is continuous at $t$.

**Exercise 15.9.** (Darboux’s Intermediate Value Theorem) Prove that if $f : (a, b) \to \mathbb{R}$ is differentiable and $a < s < t < b$, then for every $m$ strictly between $f'(s)$ and $f'(t)$ there exists $c \in (s, t)$ such that $f'(c) = m$. Hint: first assume $m = 0$; for the general case consider $g(x) = f(x) - mx$.

**Exercise 15.10.** Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{1 + e^{1/x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Find $f'(0)$. 1
Exercise 15.11. Show that
\[ \lim_{x \to \infty} \left( 1 + \frac{t}{x} \right)^x = e^t \quad \text{for all } t \in \mathbb{R}. \]

Exercise 15.12. Show that
\[ \lim_{x \to \infty} x^{1/x} = 1. \]

Exercise 15.13. Prove that if \( f \) is any rational function then
\[ \lim_{x \to \infty} f(x)e^{-x} = 0. \]

Exercise 15.14. Let \( f : (a, b) \to \mathbb{R} \) and \( t \in (a, b) \). Consider the limit
\[ (1) \quad \lim_{h \to 0} \frac{f(t + h) - f(t) - f(t - h)}{2h}. \]

(a) Show that if \( f'(t) \) exists then it equals the limit (1).
(b) Find an example where the limit (1) exists, and \( f \) is continuous and nondifferentiable at \( t \).

Exercise 15.15. Let \( f : (a, b) \to \mathbb{R} \) and \( t \in (a, b) \), and assume \( f''(t) \) exists. Prove that
\[ f''(t) = \lim_{h \to 0} \frac{f(t + h) - 2f(t) + f(t - h)}{h^2}. \]

Hint: l'Hôpital’s Rule.

Exercise 15.16. Let \( f''(a) > 0 \). Prove that there exist \( c, \delta > 0 \) such that
\[ f(x) \geq f(a) + f'(a)(x - a) + c(x - a)^2 \quad \text{if } |x - a| < \delta. \]

Hint: use l'Hôpital’s Rule to show that the fraction
\[ \frac{f(x) - f(a) - f'(a)(x - a)}{(x - a)^2} \]
has a positive limit at \( a \).