14. Consequences of Continuity

Exercise 14.1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, and assume that
\[
\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = \infty.
\]
Prove that $f$ has a minimum.

Exercise 14.2. Let $f : [0, 1] \to [0, 1]$ be continuous. Prove that there exists $c \in [0, 1]$ such that $f(c) = c$.

Exercise 14.3. Let $f : (a, b) \to (c, d)$ be increasing and onto. Prove that $f$ is continuous.

Exercise 14.4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose there exist sequences $(a_n)$ and $(b_n)$ diverging to $\infty$ such that $f(a_n) \to 0$ and $f(b_n) \to 4$. Prove that there exists a sequence $(c_j)$ such that $c_j \to \infty$ and $f(c_j) = 2$ for all $j$.

Exercise 14.5. Prove that $x^2$ is not uniformly continuous on $\mathbb{R}$.

Exercise 14.6. (a) Prove that if $f, g : A \to \mathbb{R}$ are uniformly continuous, then $f + g$ is also uniformly continuous.
   (b) Give an example where $fg$ is not uniformly continuous.

Exercise 14.7. Prove that if $f, g : A \to \mathbb{R}$ are bounded and uniformly continuous, then $fg$ is uniformly continuous.

Exercise 14.8. Prove that if $f : A \to \mathbb{R}$ is Lipschitz, that is, there exists $c \in \mathbb{R}$ such that
\[
|f(x) - f(y)| \leq c|x - y| \quad \text{for all } x, y \in \mathbb{R},
\]
then $f$ is uniformly continuous.

Exercise 14.9. Find an example of a uniformly continuous function which is not Lipschitz.

Exercise 14.10. Let $f : [0, \infty) \to \mathbb{R}$ be continuous, and assume that $\lim_{x \to \infty} f(x)$ exists. Prove that $f$ is uniformly continuous.

Exercise 14.11. Let $f : [0, 1) \to \mathbb{R}$ be continuous, and assume
\[
\lim_{x \uparrow 1} f(x)
\]
exists. Prove that $f$ is uniformly continuous.

Exercise 14.12. Prove that if $f : A \to \mathbb{R}$ is uniformly continuous and $(x_n)$ is a Cauchy sequence in $A$, then the image $(f(x_n))$ is also Cauchy.

Exercise 14.13. Give an example of $f : A \to \mathbb{R}$ which is not uniformly continuous but for which the image $(f(x_n))$ of every Cauchy sequence $(x_n)$ in $A$ is also Cauchy.

Exercise 14.14. Let $f : A \to \mathbb{R}$, and suppose $A$ is bounded. Prove that if the image $(f(x_n))$ of every Cauchy sequence $(x_n)$ in $A$ is also Cauchy, then $f$ is uniformly continuous.
Exercise 14.15. Let $f: [0,1) \to \mathbb{R}$ be uniformly continuous. Prove that $\lim_{x \to 1^-} f(x)$ exists.

Exercise 14.16. Let $f: (0,1) \to \mathbb{R}$ be uniformly continuous. Prove that $f$ is bounded.

Exercise 14.17. Let $f: A \to B$ and $g: B \to \mathbb{R}$ both be uniformly continuous. Prove that $g \circ f$ is also uniformly continuous.

Exercise 14.18. Let $f: \mathbb{R} \to \mathbb{R}$ be periodic, that is, there exists $p > 0$ such that $f(x) = f(x + p)$ for all $x \in \mathbb{R}$. Also assume that $f$ is continuous. Prove that $f$ is uniformly continuous.