13. Continuity

Exercise 13.1. In each part, give an example of a function \( f : \mathbb{R} \to \mathbb{R} \) with the indicated property:

(a) \( f \) is discontinuous everywhere;
(b) \( f \) is continuous at only one point.

Exercise 13.2. Define a sequence \((x_n)\) inductively by

\[
x_n = \begin{cases} 
2 & \text{if } n = 1 \\
\frac{x_{n-1}}{2} + \frac{1}{x_{n-1}} & \text{if } n > 1
\end{cases}
\]

Assuming the sequence \((x_n)\) converges, find its limit, and prove you are correct.

Exercise 13.3. Let \( f : A \to \mathbb{R} \) and \( t \in A \). Assume that \( f \) is discontinuous at \( t \). Prove that there exists a Cauchy sequence \((x_n)\) in \( A \) for which the image sequence \((f(x_n))\) is not Cauchy.

Exercise 13.4. Let \( f : (a, b) \to \mathbb{R} \) be monotone. Prove that \( f \) has only countably many discontinuities. Hint: for each \( x \in (a, b) \) consider the open interval with endpoints \( f(x-) \) and \( f(x+) \).

Exercise 13.5. Find an example where \( B \subset A \subset \mathbb{R} \), \( f : A \to \mathbb{R} \), and the restriction \( f|B \) is continuous, but \( f \) is not continuous on \( B \).

Exercise 13.6. Let \( A \subset \mathbb{R} \), \( t \in A \), and \( f : A \to \mathbb{R} \). We say \( f \) is continuous from the right at \( t \) if the restriction

\[ f \mid (A \cap [t, \infty)) \]

is continuous, and similarly for continuous from the left.

Prove:

(a) \( f \) is continuous from the right at \( t \) if and only if for all \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for all \( x \in A \),

\[ t \leq x < t + \delta \quad \text{then} \quad |f(x) - f(t)| < \epsilon. \]

(b) Prove that \( f \) is continuous at \( t \) if and only if it’s continuous from both the right and left at \( t \).

Exercise 13.7. Let \( f : (a, b) \to \mathbb{R} \), and assume that for each \( t \in (a, b) \) the right-hand limit \( f(t+) \) exists. Define \( g : (a, b) \to \mathbb{R} \) by \( g(t) = f(t+) \). Prove that \( g \) is continuous from the right.