3. The real numbers

Exercise 3.1. In the list of essential properties of arithmetic, show how 10–11 follow from 1–9.

Exercise 3.2. Prove by induction that $n^3 + 5n$ is divisible by 6 for all $n \in \mathbb{N}$.

Exercise 3.3. (a) Prove by induction that if $a > 1$ then the sequence $(a^n)$ is strictly increasing.
   (b) Use part (a) to prove that if $a > 1$ then $a^n > 1$ for all $n \in \mathbb{N}$.
   (c) Use part (b) to prove that if $0 < b < 1$ then $b^n < 1$ for all $n \in \mathbb{N}$.

Exercise 3.4. Let $n$ be an integer greater than 1. Prove that the function $x \mapsto x^n$ is strictly increasing on $[0, \infty)$.

Exercise 3.5. (factorials and binomial coefficients) Suppose $n$ and $k$ be nonnegative integers with $k \leq n$. Let

$$n! = \begin{cases} 1 \cdot 2 \cdots n & \text{if } n > 0 \\ 1 & \text{if } n = 0 \end{cases}$$

and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$  

Prove that if $k > 0$ then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$  

Exercise 3.6. (Binomial Theorem) Prove by induction that for all $a, b \in \mathbb{R}$ and every nonnegative integer $n$,

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$  

Exercise 3.7. (Bernoulli’s Inequality) Prove that for all $a \geq 0$ and $n \in \mathbb{N}$ we have

$$(1 + a)^n \geq 1 + na.$$