ADVANCED CALCULUS I
MAIN RESULTS

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Bolzano-Weierstrass Theorem. Every bounded sequence of real numbers has a convergent subsequence.

Continuity

*Extreme Value Theorem. Every continuous real-valued function on a closed bounded interval has both a maximum and a minimum.

*Intermediate Value Theorem. Every real-valued continuous image of an interval is an interval.

Theorem. Every continuous real-valued function on a closed bounded interval is uniformly continuous.

Theorem. Let $I$ be an interval and $f: I \to \mathbb{R}$. If $f$ is monotone and $f(I)$ is an interval then $f$ is continuous.

Theorem. Every continuous 1-1 real-valued function on an interval is strictly monotone.

Theorem. Every continuous 1-1 real-valued function on an interval has continuous inverse.

Derivatives

Chain Rule. Let $A, B \subset \mathbb{R}$, let $f: A \to B$, let $g: B \to \mathbb{R}$, and let $t \in A$. If $f$ is differentiable at $t$ and $g$ is differentiable at $f(t)$, then $g \circ f$ is differentiable at $t$ and

$$(g \circ f)'(t) = g'(f(t))f'(t).$$

*Critical Point Lemma. Let $f: (a, b) \to \mathbb{R}$ and $t \in (a, b)$. If $f$ has a max or a min at $t$, then $f'(t)$ is either 0 or does not exist.

*Mean Value Theorem. If $f: [a, b] \to \mathbb{R}$ is continuous, and differentiable on $(a, b)$, then there exists $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a).$$

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*Cauchy’s Mean Value Theorem.* If \( f, g : [a, b] \to \mathbb{R} \) are continuous, and differentiable on \((a, b)\), then there exists \( c \in (a, b) \) such that
\[
f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)).
\]

*Rolle’s Theorem.* If \( f : [a, b] \to \mathbb{R} \) is continuous, and differentiable on \((a, b)\), and if \( f(a) = f(b) \), then there exists \( c \in (a, b) \) such that \( f'(c) = 0 \).

*Taylor’s Theorem.* Let \( I \) be a closed bounded interval with endpoints \( a \) and \( b \), let \( J \) be the open interval with the same endpoints, let \( f : I \to \mathbb{R} \), and let \( n \in \mathbb{N} \). If \( f^{(n-1)} \) is continuous on \( I \) and differentiable on \( J \), then there exists \( c \in J \) such that
\[
f(b) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(b-a)^k}{k!} + \frac{f^{(n)}(c)(b-a)^n}{n!}.
\]

*Inverse Function Theorem.* Let \( I \) be an interval, \( f : I \to \mathbb{R} \), and \( t \in I \). Suppose \( f \) is 1-1 and continuous on \( I \), and differentiable at \( t \). If \( f'(t) \neq 0 \), then \( f^{-1} \) is differentiable at \( f(t) \) and
\[
(f^{-1})'(f(t)) = \frac{1}{f'(t)}.
\]

*L’Hôpital’s Rule.* Let \( f \) and \( g \) be differentiable real-valued functions on an interval. If either
\[
\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0 \quad \text{or} \quad \lim_{x \to a} g(x) = \pm \infty,
\]
then
\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},
\]
if the second limit exists.

Integrals

*Darboux’s Theorem.* Let \( f : [a, b] \to \mathbb{R} \) be integrable, and let \((u_n)\) be a sequence of Riemann sums for \( f \) associated to partitions \( P_n \) of \([a, b] \). If \( \|P_n\| \to 0 \), then \( u_n \to \int_a^b f \).

*Theorem.* Every continuous real-valued function on a closed bounded interval is integrable.

Theorem. If a bounded real-valued function on a closed bounded interval has only finitely many discontinuities, then it is integrable.

Theorem. Every monotone real-valued function on a closed bounded interval is integrable.

*Interval Additivity Theorem.* Let \( f : [a, c] \to \mathbb{R} \) and \( b \in (a, c) \). Then \( f \) is integrable on \([a, c]\) if and only if it is integrable on both \([a, b]\) and \([b, c]\), in which case
\[
\int_a^c f = \int_a^b f + \int_b^c f.
\]
Theorem. Let $I$ be an interval, $f: I \to \mathbb{R}$, and $a \in I$. Define $F: I \to \mathbb{R}$ by $F(x) = \int_a^x f$. Then $F$ is continuous.

*Mean Value Theorem for Integrals. If $f: [a, b] \to \mathbb{R}$ is continuous, then there exists $c \in [a, b]$ such that

$$\int_a^b f = f(c)(b - a).$$

Mean Value Theorem for Integrals, General Form. Let $f, g: [a, b] \to \mathbb{R}$. If $f$ is continuous and $g$ is integrable and nonnegative, then there exists $c \in [a, b]$ such that

$$\int_a^b fg = f(c) \int_a^b g.$$

*Fundamental Theorem of Calculus, Differentiating an Integral. Let $I$ be an interval, let $f: I \to \mathbb{R}$ be integrable, and let $a \in I$. Define $F: I \to \mathbb{R}$ by $F(x) = \int_a^x f$. For all $t \in I$, if $f$ is continuous at $t$, then $F$ is differentiable at $t$ and $F'(t) = f(t)$.

*Fundamental Theorem of Calculus, Integrating a Derivative. If $F: [a, b] \to \mathbb{R}$ is differentiable and $F'$ is integrable, then $\int_a^b F' = F(b) - F(a)$.

*Change of Variables Theorem. If $\phi: [a, b] \to \mathbb{R}$ is differentiable, $\phi'$ is integrable, and $f: \phi([a, b]) \to \mathbb{R}$ is continuous, then

$$\int_a^b f(\phi(x))\phi'(x) \, dx = \int_{\phi(a)}^{\phi(b)} f(u) \, du.$$

*Integration by Parts. Let $f, g: [a, b] \to \mathbb{R}$ be differentiable, and assume that $f'$ and $g'$ are integrable. Then

$$\int_a^b f'g = f(b)g(b) - f(a)g(a) - \int_a^b fg'.$$

Uniform Convergence

Theorem. Let $A \subset \mathbb{R}$, let $(f_n)$ be a uniformly convergent sequence of functions from $A$ to $\mathbb{R}$, and let $t \in A$. If each $f_n$ is continuous at $t$, then so is $\lim f_n$.

Theorem. Let $(f_n)$ be a uniformly convergent sequence of functions from $[a, b]$ to $\mathbb{R}$. If each $f_n$ is integrable, then so is $\lim f_n$, and

$$\int_a^b \lim_{n \to \infty} f_n = \lim_{n \to \infty} \int_a^b f_n.$$
Theorem. Let \( I \) be an interval, and let \((f_n)\) be a sequence of differentiable functions from \( I \) to \( \mathbb{R} \). Suppose that the sequence \((f'_n)\) of derivatives converges uniformly, and that there exists \( c \in I \) such that the sequence \((f_n(c))\) of values converges. Then \((f_n)\) converges pointwise, \(\lim_{n \to \infty} f_n\) is differentiable, and
\[
\left( \lim_{n \to \infty} f_n \right)' = \lim_{n \to \infty} f'_n.
\]

Theorem. Let \( A \subset \mathbb{R} \), let \( \sum_{n=1}^{\infty} f_n \) be a uniformly convergent series of functions from \( A \) to \( \mathbb{R} \), and let \( t \in A \). If each \( f_n \) is continuous at \( t \), then so is \( \sum_{n=1}^{\infty} f_n \).

Theorem. Let \( \sum_{n=1}^{\infty} f_n \) be a uniformly convergent series of functions from \([a, b]\) to \( \mathbb{R} \). If each \( f_n \) is integrable, then so is \( \sum_{n=1}^{\infty} f_n \), and
\[
\int_a^b \sum_{n=1}^{\infty} f_n(x)\, dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\, dx.
\]

Theorem. Let \( I \) be an interval, and let \( \sum_{n=1}^{\infty} f_n \) be a series of differentiable functions from \( I \) to \( \mathbb{R} \). Suppose that the series \( \sum_{n=1}^{\infty} f'_n \) of derivatives converges uniformly, and that there exists \( c \in I \) such that the series \( \sum_{n=1}^{\infty} f_n(c) \) of values converges. Then \( \sum_{n=1}^{\infty} f_n \) converges pointwise, \( \sum_{n=1}^{\infty} f_n \) is differentiable, and
\[
\left( \sum_{n=1}^{\infty} f_n \right)' = \sum_{n=1}^{\infty} f'_n.
\]

Weierstrass M-Test. Let \( A \subset \mathbb{R} \), and let \( \sum_{n=1}^{\infty} f_n \) be a series of functions from \( A \) to \( \mathbb{R} \). Suppose that there exists a convergent series \( \sum_{n=1}^{\infty} M_n \) of nonnegative real numbers such that for all \( n \in \mathbb{N} \) and \( x \in A \) we have \( |f_n(x)| \leq M_n \). Then \( \sum_{n=1}^{\infty} f_n \) converges uniformly.

Power Series

Cauchy-Hadamard Theorem. For every power series \( \sum_{n=0}^{\infty} c_n(x - a)^n \), the radius of convergence exists and equals
\[
r := \frac{1}{\limsup |c_n|^{1/n}},
\]
interpreted as 0 if the lim sup is \( \infty \), and \( \infty \) if the lim sup is 0.

Theorem. If the power series \( \sum_{n=0}^{\infty} c_n(x - a)^n \) has radius of convergence \( r \), then it converges absolutely on \((a - r, a + r)\).
*Theorem.* If the power series \( \sum_{n=0}^{\infty} c_n(x - a)^n \) has radius of convergence \( r \), then both power series
\[
\sum_{n=1}^{\infty} nc_n(x - a)^{n-1} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x - a)^{n+1}
\]
also have radius of convergence \( r \).

Moreover, if \( r > 0 \) and \( f: (a - r, a + r) \to \mathbb{R} \) is defined by \( f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n \), then for all \( x \in (a - r, a + r) \) we have
\[
f'(x) = \sum_{n=1}^{\infty} nc_n(x - a)^{n-1} \quad \text{and} \quad \int_a^x f = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x - a)^{n+1}.
\]

*Theorem.* If \( f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n \) for all \( x \) in a nonempty open interval containing \( a \), then for all \( n = 0, 1, \ldots \) we have
\[
c_n = \frac{f^{(n)}(a)}{n!}.
\]

**Abel’s Theorem.** If the power series \( \sum_{n=0}^{\infty} c_n(x - a)^n \) converges at \( b \), then it converges uniformly on the closed interval with endpoints \( a \) and \( b \).