Our goal is to prove the following results:

**Critical Point Lemma.** Let \( f : (a, b) \to \mathbb{R} \) and \( t \in (a, b) \). If \( f \) has a max or a min at \( t \), then \( f'(t) \) is either 0 or does not exist.

**Mean Value Theorem.** If \( f : [a, b] \to \mathbb{R} \) is continuous, and differentiable on \((a, b)\), then there exists \( c \in (a, b) \) such that
\[
  f(b) - f(a) = f'(c)(b - a).
\]

**Cauchy’s Mean Value Theorem.** If \( f, g : [a, b] \to \mathbb{R} \) are continuous, and differentiable on \((a, b)\), then there exists \( c \in (a, b) \) such that
\[
  f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)).
\]

**Rolle’s Theorem.** If \( f : [a, b] \to \mathbb{R} \) is continuous, and differentiable on \((a, b)\), and if \( f(a) = f(b) \), then there exists \( c \in (a, b) \) such that \( f'(c) = 0 \).

**Taylor’s Theorem.** Let \( I \) be a closed bounded interval with endpoints \( a \) and \( b \), let \( J \) be the open interval with the same endpoints, let \( f : I \to \mathbb{R} \), and let \( n \in \mathbb{N} \). If \( f^{(n-1)} \) is continuous on \( I \) and differentiable on \( J \), then there exists \( c \in J \) such that
\[
  f(b) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(b-a)^k}{k!} + \frac{f^{(n)}(c)(b-a)^n}{n!}.
\]

*Proof of Critical Point Lemma.* Without loss of generality assume \( f \) has a maximum at \( t \); to handle the other case take negatives. For all \( x \in (a, t) \) we have
\[
  \frac{f(x) - f(t)}{x - t} \geq 0,
\]
and letting \( x \uparrow t \) we get \( f'(t) \geq 0 \). On the other hand, for all \( x \in (t, b) \) we have
\[
  \frac{f(x) - f(t)}{x - t} \leq 0,
\]
and letting \( x \downarrow t \) we get \( f'(t) \leq 0 \). Therefore, we must have \( f'(t) = 0 \). \(\text{QED}\)

*Date:* September 23, 2005.
**Proof of Rolle’s Theorem.** By the Extreme Value Theorem, \( f \) has a maximum and a minimum on \([a, b]\). Since \( f(a) = f(b) \), at least one of the maximum or minimum must occur at some \( c \in (a, b) \) (even if \( f \) is constant). By the Critical Point Lemma, \( f'(c) = 0 \). \( \text{QED} \)

**Proof of Cauchy’s Mean Value Theorem.** Define \( h : [a, b] \to \mathbb{R} \) by 
\[
h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a)).
\] Then \( h \) satisfies the hypotheses of Rolle’s Theorem, so there exists \( c \in (a, b) \) such that 
\[
0 = h'(c) = f'(c)(g(b) - g(a)) - g'(c)(f(b) - f(a)),
\] and this implies the desired equation. \( \text{QED} \)

**Proof of Mean Value Theorem.** Just apply Cauchy’s Mean Value Theorem with \( g(x) = x \). \( \text{QED} \)

**Proof of Taylor’s Theorem.** Define \( g : I \to \mathbb{R} \) by 
\[
g(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(x-a)^k}{k!} + M(x-a)^n - f(x),
\] where \( M \in \mathbb{R} \) is chosen such that \( g(b) = 0 \). Note that \( g^{(k)}(a) = 0 \) for \( k = 0, \ldots, n-1 \). Since \( g(b) = 0 \), by Rolle’s Theorem there exists \( c_1 \) strictly between \( a \) and \( b \) such that \( g'(c_1) = 0 \). Then since \( g'(a) = 0 \), again by Rolle’s Theorem there exists \( c_2 \) strictly between \( a \) and \( c_1 \) such that \( g''(c_2) = 0 \). Continuing inductively, after \( n \) steps we get \( c := c_n \) strictly between \( a \) and \( c_{n-1} \) such that \( g^{(n)}(c) = 0 \). Since \( g^{(n)}(x) = Mn! - f^{(n)}(x) \), we get 
\[
M = \frac{f^{(n)}(c)}{n!}.
\] Coupled with \( g(b) = 0 \), this gives the desired result. \( \text{QED} \)