Our goal is to prove the following results:

**Theorem 1.** Every continuous real-valued function on a closed bounded interval is integrable.

**Theorem 2.** If a bounded real-valued function on a closed bounded interval has only finitely many discontinuities, then it is integrable.

**Theorem 3.** Every monotone real-valued function on a closed bounded interval is integrable.

We will need the following auxiliary result:

**Lemma.** Let \( f : [a,b] \to \mathbb{R} \) be bounded, and assume that for every \( t \in (a,b) \) the function \( f \) is integrable on \([a,t]\). Then \( f \) is integrable on \([a,b]\). Similarly if \( f \) is integrable on \([t,b]\) for every \( t \in (a,b)\).

We give the proofs of the theorems, after which we prove the lemma:

**Proof of Theorem 1.** Let \( f : [a,b] \to \mathbb{R} \) be continuous. Since the interval \([a,b]\) is closed and bounded, \( f \) is uniformly continuous. Let \( \epsilon > 0 \). Choose \( \delta > 0 \) such that for all \( x, y \in [a,b] \), if \( |x - y| < \delta \) then

\[
|f(x) - f(y)| < \frac{\epsilon}{2(b - a)}.
\]

Now choose a partition \( P = \{x_1, \ldots, x_n\} \) of \([a,b]\) such that \( \|P\| < \delta \). Then for all \( i = 1, \ldots, n \) and all \( x, y \in [x_{i-1}, x_i] \) we have \( |f(x) - f(y)| < \epsilon/(2(b - a)) \). Thus for all \( i = 1, \ldots, n \) we have

\[
M_i - m_i \leq \frac{\epsilon}{2(b - a)}.
\]
Hence

\[ U(P) - L(P) = \sum_{i=1}^{n} (M_i - m_i) \Delta x_i \leq \frac{\epsilon}{2(b - a)} \sum_{i=1}^{n} \Delta x_i \]

\[ = \frac{\epsilon(b - a)}{2(b - a)} = \frac{\epsilon}{2} < \epsilon. \]

Therefore \( f \) is integrable. \( \text{QED} \)

**Proof of Theorem 2.** There is a partition of the interval so that on each subinterval \( f \) is continuous except possibly at one endpoint. By the Interval Additivity Theorem, it suffices to show that \( f \) is integrable on each of these subintervals. By symmetry, it suffices to assume that \( f : [a, b] \to \mathbb{R} \) is bounded, and continuous except at \( b \). Then by Theorem 1, for every \( t \in (a, b) \) the function \( f \) is integrable on \( [a, t] \), hence is integrable on \([a, b]\) by the lemma. \( \text{QED} \)

**Proof of Theorem 3.** Let \( f : [a, b] \to \mathbb{R} \) be monotone. Without loss of generality \( f \) is increasing; obvious changes handle the case where \( f \) is decreasing. Then for any partition \( P = \{x_1, \ldots, x_n\} \) of \([a, b]\),

\[ M_i = \max\{f(x) : x_{i-1} \leq x \leq x_i\} = f(x_i) \]
\[ m_i = \min\{f(x) : x_{i-1} \leq x \leq x_i\} = f(x_{i-1}). \]

Given \( \epsilon > 0 \), choose \( n \in \mathbb{N} \) such that

\[ \frac{(b - a)(f(b) - f(a))}{n} < \epsilon. \]

Define a partition \( P \) of \([a, b]\) by

\[ x_i = a + i\left(\frac{b - a}{n}\right) \quad \text{for } i = 0, 1, \ldots, n. \]
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Then

\[ U(P) - L(P) = \sum_{i=1}^{n} (M_i - m_i) \Delta x_i \]

\[ = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \frac{b-a}{n} \]

\[ = \frac{b-a}{n} (f(x_n) - f(x_0)) \quad \text{(telescoping sum)} \]

\[ = \frac{b-a}{n} (f(b) - f(a)) \]

\[ < \epsilon. \]

Thus \( f \) is integrable. \quad \text{QED}

Proof of Lemma. We prove the first statement; the second statement then follows by symmetry. So, assume that for every \( t \in (a, b) \) the function \( f \) is integrable on \([a, t]\). Let \( M = \sup \{|f(x)| : x \in [a, b]\} \). Given \( \epsilon > 0 \), choose \( t \in (a, b) \) such that \( 2M(b - t) < \epsilon/2 \). Since \( f \) is integrable on \([a, t]\) by hypothesis, there exists a partition \( P \) of \([a, t]\) such that \( U(P) - L(P) < \epsilon/2 \). Put \( R = \{t, b\} \) and \( Q = P \cup R \). Then \( R \) is a partition of \([t, b]\) and \( Q \) is a partition of \([a, b]\). For all \( x, y \in [t, b] \) we have \( |f(x) - f(y)| \leq 2M \). Thus

\[ U(Q) - L(Q) = U(P) - L(P) + U(R) - L(R) \]

\[ \leq \frac{\epsilon}{2} + 2M(b - t) \]

\[ < \epsilon. \]

Thus \( f \) is integrable on \([a, b]\). \quad \text{QED}