Our goal is to prove the following results:

**Fundamental Theorem of Calculus, Differentiating an Integral.** Let $I$ be a closed bounded interval, let $f: I \to \mathbb{R}$ be integrable, and let $a \in I$. Define $F: I \to \mathbb{R}$ by $F(x) = \int_a^x f$. For all $t \in I$, if $f$ is continuous at $t$, then $F$ is differentiable at $t$ and $F'(t) = f(t)$.

**Fundamental Theorem of Calculus, Integrating a Derivative.** If $F: [a,b] \to \mathbb{R}$ is differentiable and $F'$ is integrable, then $\int_a^b F' = F(b) - F(a)$.

**Proof of Differentiating an Integral.** If $x \in I \setminus \{t\}$ then

\[
\left| \frac{F(x) - F(t)}{x-t} - f(t) \right| = \left| \frac{\int_t^x f - \int_t f}{x-t} - f(t) \right| \\
= \left| \frac{\int_t^x f(s) ds}{x-t} - \frac{\int_t^x f(t) ds}{x-t} \right| \\
= \left| \frac{\int_t^x (f(s) - f(t)) ds}{x-t} \right| \\
= \left| \frac{\int_t^x (f(s) - f(t)) ds}{|x-t|} \right|.
\]

Let $\epsilon > 0$. Choose $\delta > 0$ such that for all $x \in I$, if $|x-t| < \delta$ then $|f(x) - f(t)| < \epsilon/2$. Let $x \in I \setminus \{t\}$. Assume that $|x-t| < \delta$. Then for all $s$ between $x$ and $t$ we have $|f(s) - f(t)| < \epsilon/2$. Thus

\[
\left| \int_t^x (f(s) - f(t)) ds \right| \leq \left| \int_t^x f(s) ds \right| - \left| \int_t^x f(t) ds \right| \leq \frac{\epsilon}{2} \left| \int_t^x ds \right| = \frac{\epsilon |x-t|}{2},
\]

hence

\[
\left| F(x) - F(t) \right| \leq \frac{\epsilon}{2} < \epsilon.
\]

Thus $F'(t) = f(t)$. QED
Proof of Integrating a Derivative. Let \( \epsilon > 0 \). Choose a partition \( P = \{x_0, \ldots, x_n\} \) of \([a, b]\) such that for every associated Riemann sum \( u \) we have \( |u - \int_a^b F'| < \epsilon \). By the Mean Value Theorem (for derivatives), for each \( i = 1, \ldots, n \) there exists \( t_i \in (x_{i-1}, x_i) \) such that

\[
F(x_i) - F(x_{i-1}) = F'(t_i) \Delta x_i.
\]

Thus

\[
F(b) - F(a) = \sum_{i=1}^n (F(x_i) - F(x_{i-1})) = \sum_{i=1}^n F'(t_i) \Delta x_i,
\]

so \( |F(b) - F(a) - \int_a^b F'| < \epsilon \). Since \( \epsilon > 0 \) was arbitrary, we have

\[
F(b) - F(a) - \int_a^b F' = 0,
\]

hence

\[
\int_a^b F' = F(b) - F(a).
\]

QED